Endomorphism Breaking in Graphs

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Abstract

We introduce the *endomorphism distinguishing number* $D_e(G)$ of a graph G as the least cardinal d such that G has a vertex coloring with d colors that is only preserved by the trivial endomorphism. This generalizes the notion of the distinguishing number $D(G)$ of a graph G, which is defined for automorphisms instead of endomorphisms.

As the number of endomorphisms can vastly exceed the number of automorphisms, the new concept opens challenging problems, several of which are presented

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here. In particular, we investigate relationships between $D_e(G)$ and the endomorphism motion of a graph. Moreover, we extend numerous results about the distinguishing number of finite and infinite graphs to the endomorphism distinguishing number. This is the main concern of the paper.

Keywords: Distinguishing number; Endomorphisms; Infinite graphs;

1 Introduction

Albertson and Collins [1] introduced the *distinguishing number* $D(G)$ of a graph G as the least cardinal d such that G has a labeling with d labels that is only preserved by the trivial automorphism.

This concept has spawned numerous papers, mostly on finite graphs. But countable infinite graphs have also been investigated with respect to the distinguishing number; see [9], [15], [16], and [17]. For graphs of higher cardinality compare [10].

The aim of this paper is the presentation of fundamental results for the endomorphism distinguishing number, and of open problems. In particular, we extend the Motion Lemma of Russell and Sundaram [14] to endomorphisms, present an Endomorphism Motion Conjecture, which generalizes the Infinite Motion Conjecture of [4], prove the validity of two special cases, and support the conjecture by examples.

2 Definitions and Basic Results

As the distinguishing number has already been defined, let us note that $D(G) = 1$ for all asymmetric graphs. This means that almost all finite graphs have distinguishing number 1, because almost all graphs are asymmetric, see Erdős and Rényi [5]. Clearly $D(G) \geqslant 2$ for all other graphs. Again, it is natural to conjecture that almost all of them have distinguishing number 2. This is supported by the observations of Conder and Tucker [3].

However, for the complete graph K_n, and the complete bipartite graph $K_{n,n}$ we have $D(K_n) = n$, and $D(K_{n,n}) = n + 1$. Furthermore, the distinguishing number of the cycle of length 5 is 3, but cycles C_n of length $n \geqslant 6$ have distinguishing number 2.

This compares with a more general result of Klavžar, Wong and Zhu [12] and of Collins and Trenk [2], which asserts that $D(G) \leqslant \Delta + 1$, equality holding if and only if G is a K_n, $K_{n,n}$ or C_5.

Now to the endomorphism distinguishing number. Before defining it, let us recall that an *endomorphism* of a graph $G = (V, E)$ is a mapping $\phi : V \to V$ such that for every edge $uv \in E$ its image $\phi(u)\phi(v)$ is an edge, too.

Definition *The endomorphism distinguishing number $D_e(G)$ of a graph G is the least cardinal d such that G has a labeling with d labels that is preserved only by the identity endomorphism of G.*

Let us add that we also say *colors* instead of *labels*. If a labeling c is not preserved by an endomorphism ϕ, we say that *c breaks ϕ*.

Clearly $D(G) \leqslant D_e(G)$. For graphs G with $\mathrm{Aut}(G) = \mathrm{End}(G)$ equality holds. Such graphs are called *core graphs*. Notice that complete graphs and odd cycles are core graphs, see [7]. Hence $D_e(K_n) = n$, $D_e(C_5) = 3$, and $D_e(C_{2k+1}) = 2$ for $k \geqslant 3$.

Interestingly, almost all graphs are core graphs, as shown by Koubek and Rödl [13]. Because almost all graphs are asymmetric, this implies that almost all graphs have trivial endomorphism monoid, that is, $\mathrm{End}(G) = \{\mathrm{id}\}$. Graphs with trivial endomorphism monoid are called *rigid*. Clearly $D_e(G) = 1$ for any rigid graph G, and thus $D_e(G) = 1$ for almost all graphs G.

$D_e(G)$ can be equal to $D(G)$ even when $\mathrm{Aut}(G) \subsetneq \mathrm{End}(G)$. For example, this is the case for even cycles. We formulate this as a lemma.

Lemma 1 *The automorphism group of even cycles is properly contained in their endomorphism monoid, but $D(C_{2k}) = D_e(C_{2k})$ for all $k \geqslant 2$.*

Proof. It is easily seen that every even cycle admits proper endomorphisms, that is, endomorphisms that are not automorphisms. Furthermore, it is readily verified that $D(C_4) = D_e(C_4)$.

Hence, let $k \geqslant 3$. Color the vertices v_1, v_2 and v_4 black and all other vertices white, see Figure 1. We wish to show that this coloring is endomorphism distinguishing. Clearly this coloring distinguishes all automorphisms.

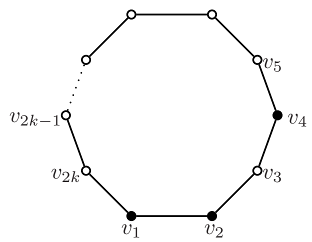

Figure 1: Distinguishing an even cycle

Let ϕ be a proper endomorphism. It has to map the cycle into a proper subgraph of itself. Thus, $\phi(C_{2k})$ must be a path, say P.

Furthermore, all edges with only white, resp. black, endpoints must be mapped into edges that have only white, resp. black, endpoints. Hence v_1v_2 is mapped into itself. Because $v_{2k-1}v_{2k}$ is the only edge with two white endpoints that is adjacent to v_1v_2, it must also be mapped into itself. This fixes v_{2k-1}, v_{2k}, v_1 and v_2. But then v_3 and v_4 are also fixed.

Now we observe that the path $v_4v_5 \cdots v_{2k}v_1$ in C_{2k} has only white interior vertices and that it it has to be mapped into a walk in P from v_4 to v_1 that contains only white interior vertices. Clearly this is not possible. $\square$

To show that $D(G)$ can be smaller than $D_e(G)$, we consider graphs G with trivial automorphism group but nontrivial endomorphisms monoid. For such graphs $D(G) = 1$, but $D_e(G) > 1$. Easy examples are asymmetric, nontrivial trees T. For, every such tree has at least 7 vertices and at least three vertices of degree 1. Let a be a vertex of degree 1 and b its neighbor. Because T has at least 7 vertices and since it is connected, there must be a neighbor c of b that is different from s. Then the mapping

$$\phi : v \mapsto \begin{cases} c \text{ if } v = a \\ v \text{ otherwise} \end{cases}$$

is a nontrivial endomorphism.

3 The Endomorphism Motion Lemma

Russel and Sundaram [14] proved that the distinguishing number of a graph is small when every automorphism of G moves many elements. We generalize this result to endomorphisms and begin with the definition of motion.

The *motion* $m(\phi)$ of a nontrivial endomorphism ϕ of a graph G, is the number of elements it moves:

$$m(\phi) = |\{v \in V(G) \,|\, \phi(v) \neq v\}|.$$

The *endomorphism motion* of a graph G is

$$m_e(G) = \min_{\phi \in \mathrm{End}(G) \setminus \{\mathrm{id}\}} m(\phi)$$

For example, $m_e(C_4) = 1, m_e(C_5) = 4, m_e(C_{100}) = 49, m_e(K_{100}) = 2$.

In the sequel we will prove the following generalization of Theorem 1 of Russell and Sundaram [14].

Lemma 2 (Endomorphism Motion Lemma) *For any graph G*

$$d^{\frac{m_e(G)}{2}} \geqslant |\,\mathrm{End}(G)| \tag{1}$$

implies $D_e(G) \leqslant d$.

To prepare for the proof we define orbits of endomorphisms.

Definition *An orbit of an endomorphism ϕ of a graph G is an equivalence class with respect to the equivalence relation $\sim$ on $V(G)$, where $u \sim v$ if there exist nonnegative integers i and j such that $\phi^i(u) = \phi^j(v)$.*

The orbits form a partition $V(G) = I_1 \cup I_2 \cup \cdots \cup I_k$, $I_i \cap I_j = \emptyset$ for $1 \leqslant i < j \leqslant k$, of $V(G)$. For finite graphs it can be characterized as the unique partition with the maximal number of sets that is invariant under ϕ^{-1}. For infinite graphs we characterize it as the finest partition that is invariant under ϕ^{-1}. For automorphisms it coincides with the cycle decomposition.

The *orbit norm* of an endomorphism ϕ with the orbits $I_1, I_2, \ldots, I_k$ is

$$o(\phi) = \sum_{i=1}^{k} (|I_i| - 1)\,,$$

and the *endomorphism orbit norm* of a graph G is

$$o(G) = \min_{\phi \in \mathrm{End}(G) \setminus \{\mathrm{id}\}} o(\phi)\,.$$

Notice that ϕ may not move all elements of a nontrivial orbit, whereas automorphisms move all elements in a nontrivial cycle of the cycle decomposition. To see this, consider an orbit $I = \{a, b\}$, where $\phi(a) = b$, and $\phi(b) = b$. Only one element of the orbit is moved, and the contribution of I to the orbit norm of ϕ is 1. Clearly $o(\phi) \geqslant m(\phi)/2$, and thus $o(G) \geqslant m_e(G)/2$.

Proof of Lemma 2 Let $n = |V(G)|$. For any nonidentity $\phi \in \mathrm{End}(G)$, the number of d-colorings preserved by ϕ is $d^{o(\phi)}$, because each orbit must get the same color. There are $n - m(\phi)$ singleton orbits and the rest falls into at most $m(\phi)/2$ orbits. Thus

$$o(\phi) \leqslant n - m(\phi) + m(\phi)/2 \leqslant n - m_e(G)/2.$$

We conclude that the number of d-colorings that are preserved by all endomorphisms is at most $(|\mathrm{End}(G)|-1)\, d^{n-m_e(G)/2}$. If $d^{m_e(G)/2} > |\mathrm{End}(G)|-1$, then the number of d-colorings preserved is less than the total number d^n of d-colorings. Thus if $d^{m_e(G)/2} \geqslant |\mathrm{End}(G)|$, we have $D_e(G) = d$. □

This proof is an adaptation of the proof in [11] of the Motion Lemma for the distinguishing number. Of course, Lemma 2 is also an easy consequence of the Orbit Norm Lemma.

Lemma 3 (Orbit Norm Lemma) *A graph G is endomorphism d-distinguishable if*

$$\sum_{\phi \in \mathrm{End}(G) \setminus \{\mathrm{id}\}} d^{-o(\phi)} < 1.$$

Proof. We study the behavior of a random d-coloring c of G, the probability distribution given by selecting the color of each vertex independently and uniformly in the set $\{1, \ldots, d\}$. Fix an endomorphism $\phi \neq \mathrm{id}$ and consider the bad event that the random coloring c is preserved by ϕ, that is, $c(v) = c(\phi(v))$ for each vertex v of G. Then it is easily seen that

$$\mathrm{Prob}_c[\forall v :\ c(v) = c(\phi(v))] = \left(\frac{1}{d}\right)^{o(\phi)} \leqslant \left(\frac{1}{d}\right)^{o(G)}.$$

Collecting together these bad events, we have

$$\mathrm{Prob}_c[\exists\phi \neq \mathrm{id}\forall v:\, c(v) = c(\phi(v))] \leqslant \sum_{\phi\in \mathrm{End}(G)\setminus\{\mathrm{id}\}} \left(\frac{1}{d}\right)^{o(\phi)}.$$

By hypothesis the left side is less than one, thus there exists a coloring c such that for all nontrivial ϕ there is a v, such that $c(v) \neq c(\phi(v))$, as desired. □

Lemma 2 compares with Theorem 1 of Russell and Sundaram [14]. It asserts that G is 2-distinguishable if

$$m(G) > 2\log_2 |\mathrm{Aut}(G)|\,,$$

where

$$m(G) = \min_{\phi\in\mathrm{Aut}(G)\setminus\{\mathrm{id}\}} m(\phi)\,.$$

Actually Russell and Sundaram prove that G is d-distinguishable under the assumption $m(G) > 2\log_d |\mathrm{Aut}(G)|$. Furthermore, a close look at their proof shows that it suffices to require that

$$m(G) \geqslant 2\log_d |\mathrm{Aut}(G)|. \tag{2}$$

Thus, our Endomorphism Motion Lemma is a direct generalization of their result that Equation 2 implies d-distinguishability. We will refer to it as the *Motion Lemma*, or the *Motion Lemma of Russell and Sundaram.*

The Motion Lemma allows to compute the distinguishing number of many classes of finite graphs. We know of no such applications for the Endomorphism Motion Lemma, but will show the applicability of its generalization to infinite graphs.

4 Infinite graphs

Suppose we are given an infinite graph G with infinite endomorphism motion $m_e(G)$ and wish to generalize Equation 1 to this case for finite d. Notice that

$$d^{m_e(G)/2} = d^{m_e(G)} = 2^{m_e(G)}$$

in this situation. Thus the natural generalization would be that

$$2^{m_e(G)} \geqslant |\,\mathrm{End}(G)| \tag{3}$$

implies endomorphism 2-distinguishabilty. We formulate this as a conjecture.

Endomorphism Motion Conjecture. *Let G be a connected, infinite graph with endomorphism motion $m_e(G)$. If $2^{m_e(G)} \geqslant |\mathrm{End}(G)|$, then $D_e(G) = 2$.*

Let us consider the case where G is countable first. If $m_e(G)$ is infinite, then $m_e(G) = \aleph_0$ and $2^{m_e(G)} = 2^{\aleph_0} = \mathfrak{c}$, where $\mathfrak{c}$ denotes he cardinality of the continuum.

Furthermore, for countable graphs we have $|\operatorname{End}(G)| \leqslant \aleph_0^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. This means that Equation 3 is always satisfied for countably infinite graphs with infinite motion. This motivates the following conjecture.

Infinite Endomorphism Motion Conjecture. *Let G be a countable connected graph with infinite endomorphism motion. Then G is endomorphism 2-distinguishable.*

In the last section we will verify this conjecture for countable trees with infinite endomorphism motion. Their endomorphism monoids are uncountable and we will see that they have endomorphism distinguishing number 2.

We now show the validity of the conjecture for countable endomorphism monoids. In fact, we can show an even stronger result, namely that in the case of graphs with countable endomorphism monoid almost every coloring is distinguishing.

Theorem 4 *Let G be a graph with infinite motion whose endomorphism monoid is countable. Let c be a random 2-coloring where all vertices have been colored independently and assume that there is an $\varepsilon > 0$ such that for every vertex v the probability that it is assigned the color $x \in \{black, white\}$ satisfies*

$$\varepsilon \leqslant \operatorname{Prob}[c(v) = x] \leqslant 1 - \varepsilon.$$

Then c is almost surely distinguishing.

Proof. First, let $\phi \in \operatorname{End}(G)$ be a fixed endomorphism of G. Since the motion of ϕ is infinite we can find infinitely many disjoint pairs $\{v_i, \phi(v_i)\}$. Clearly the colorings of these pairs are independent and the probability that ϕ preserves the coloring in any of the pairs is bounded from above by some constant $\varepsilon' < 1$. Now

$$\operatorname{Prob}[\phi \text{ preserves } c] \leqslant \operatorname{Prob}[\forall i\colon c(v_i) = c(\phi(v_i))] = 0.$$

Since there are only countably many endomorphisms we can use σ-subadditivity of the probability measure to conclude that

$$\operatorname{Prob}[\exists \phi \in \operatorname{End}(G)\colon \phi \text{ preserves c}] \leqslant \sum_{\phi \in \operatorname{End}(G)} \operatorname{Prob}[\phi \text{ preserves } c] = 0$$

which completes the proof. □

We will usually only use the following Corollary of Theorem 4.

Corollary 5 *Let G be a graph with infinite motion whose endomorphism monoid is countable. Then*

$$D_e(G) = 2.$$

An analogue of the endomorphism motion conjecture for countable structures for the distinguishing number is

Infinite Motion Conjecture of Tucker [15]. *Let G be a connected, locally finite infinite graph with infinite motion. Then G is 2-distinguishable.*

The same proof as the one of Theorem 4 shows that the conjecture is true if $\mathrm{Aut}(G)$ is countable, see [4]. There are numerous applications of this result, see [11].

For the Infinite Endomorphism Motion Conjecture we have the following theorem. It is an immediate generalization of [10, Theorem 3.2].

Theorem 6 *Let Γ be a finitely generated infinite group. Then there is a 2-coloring of the elements of Γ, such that the identity endomorphism of Γ is the only endomorphism that preserves this coloring. In other words, finitely generated groups are endomorphism 2-distinguishable.*

Proof. Let S be a finite set of generators of Γ that is closed under inversion. Since every element g of Γ can be represented as a product $s_1 s_2 \cdots s_k$ of finite length in elements of S, we infer that Γ is countable.

Also, if $\phi \in \mathrm{End}(\Gamma)$, then

$$\phi(g) = \phi(s_1 s_2 \cdots s_k) = \phi(s_1)\phi(s_2) \cdots \phi(s_k).$$

Hence, every endomorphism ϕ is determined by the finite set

$$\phi(S) = \{\phi(s) \mid s \in S\}.$$

Because every $\phi(s)$ is a word of finite length in elements of S there are only countably many elements in $\phi(S)$. Hence $\mathrm{End}(\Gamma)$ is countable.

Now, let us consider the motion of the nonidentity elements of $\mathrm{End}(\Gamma)$. Let ϕ be such an element and consider the set

$$\mathrm{Fix}(\phi) = \{g \in \Gamma \mid \phi(g) = g\}.$$

It is easily seen that these elements form a subgroup of Γ. Since ϕ does not fix all elements of Γ it is a proper subgroup. Since its smallest index is two, the set $\Gamma \setminus \mathrm{Fix}(\phi)$ is infinite. Thus $m(\phi)$ is infinite. As ϕ was arbitrarily chosen, Γ has infinite endomorphism motion.

By Corollary 5 we conclude that Γ is 2-distinguishable. $\square$

The next theorem shows that the endomorphism motion conjecture is true if $m_e(G) = |\mathrm{End}(G)|$, even if $m_e(G)$ is not countable.

Theorem 7 *Let G be a connected graph with uncountable endomorphism motion. Then $|\mathrm{End}(G)| \leqslant m_e(G)$ implies $D_e(G) = 2$.*

Proof. Set $\mathfrak{n} = |\mathrm{End}(G)|$, and let ζ be the smallest ordinal number whose underlying set has cardinality $\mathfrak{n}$. Furthermore, choose a well ordering $\prec$ of $A = \mathrm{End}(G) \setminus \{\mathrm{id}\}$ of order type ζ, and let ϕ_0 be the smallest element with respect to $\prec$. Then the cardinality of the set of all elements of A between ϕ_0 and any other $\phi \in A$ is smaller than $\mathfrak{n} \leqslant m_e(G)$.

Now we color all vertices of G white and use transfinite induction to break all endomorphisms by coloring selected vertices black. By the assumptions of the theorem, there exists a vertex v_0 that is not fixed by ϕ_0. We color it black. This coloring breaks ϕ_0.

For the induction step, let $\psi \in A$. Suppose we have already broken all $\phi \prec \psi$ by pairs of vertices $(v_\phi, \phi(v_\phi))$, where v_ϕ and $\phi(v_\phi)$ have distinct colors. Clearly, the cardinality of the set R of all $(v_\phi, \phi(v_\phi))$, $\phi \prec \psi$, is less than $\mathfrak{n} \geqslant m_e(G)$. By assumption, ψ moves at least $m_e(G)$ vertices. Since there are still $\mathfrak{n}$ vertices not in R, there must be a vertex v_ψ that does not meet R. If $\psi(v_\psi)$ is white, we color v_ψ black. Otherwise, we color it white. This coloring breaks ψ. $\square$

Corollary 8 *Let G be a connected graph with uncountable endomorphism motion. If the general continuum hypothesis holds, and if $|\mathrm{End}(G)| < 2^{m_e(G)}$, then $D_e(G) = 2$.*

Proof. By the generalized continuum hypothesis $2^{m_e(G)}$ is the successor of $m_e(G)$. Hence, the inequality $2^{m_e(G)} > |\mathrm{End}(G)|$ is equivalent to $m_e(G) \geqslant |\mathrm{End}(G)|$. $\square$

5 Examples and outlook

So far we have only determined the endomorphism distinguishing numbers of core graphs, such as the complete graph and odd cycles, and proved that $D_e(C_{2k}) = 2$ for $k \geqslant 3$. Furthermore, it is easily seen that $D_e(K_{n,n}) = n+1$ and $D_e(K_{m,n}) = \max(m,n)$ if $m \neq n$.

In the case of infinite structures we proved Theorem 6, which shows that $D_e(\Gamma) = 2$ for finitely generated, infinite groups Γ.

We will now determine the endomorphism distinguishing numbers of finite and infinite paths and we begin with the following lemma.

Lemma 9 *Let ϕ be an endomorphism of a (possibly infinite) tree G such that $\phi(u) = \phi(v)$ for two distinct vertices u, v. Then there exist two vertices x, y on the path between u and v such that $\phi(x) = \phi(y)$ and $\mathrm{dist}(x, y) = 2$.*

Proof. Suppose $\mathrm{dist}(u, v) \neq 2$. Hence $\mathrm{dist}(u, v) > 2$. Let P be the path connecting u and v in G, and let P' be the subgraph induced by the image $\phi(P)$. Clearly, P' is a finite tree with at least one edge.

Because every nontrivial finite tree has at least two pendant vertices, there must be a pendant vertex w of P' that is different from $\phi(u) = \phi(v)$. Thus $w = \phi(z)$ for some internal vertex z of P. If x and y are the two neighbors of z on P, then clearly $\phi(x) = \phi(y)$ and $\mathrm{dist}(x, y) = 2$. $\square$

The above lemma implies the following corollary for finite graphs, because any injective endomorphism of a finite graph is an automorphism.

Corollary 10 *Let G be a finite tree. Then for every $\phi \in$End$(G)\setminus$Aut(G) there exist two vertices x, y of distance 2 such that $\phi(x) = \phi(y)$.* $\square$

Lemma 11 *The endomorphism distinguishing number of all finite paths P_n of order $n \geqslant 2$ is two.*

Proof. Clearly, $D_e(P_n) \geqslant 2$ since $\mathrm{End}(P_n) \neq \mathrm{Aut}(P_n)$. To see that $D_e(P_n) = 2$ consider the following labeling

$$c(P_n) = \begin{cases} (11221122.....1122) & \text{if } n \equiv 0 \mod 4 \\ (11221122...11221) & \text{if } n \equiv 1 \mod 4 \\ (1221122.....22112) & \text{if } n \equiv 2 \mod 4 \\ (11221122...22112) & \text{if } n \equiv 3 \mod 4 \end{cases}.$$

The only nontrivial automorphism of a path (symmetry with respect to the center) does not preserve this labeling. By Corollary 10, any other endomomorphism $\phi \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$ has to identify two vertices of distance two. Then ϕ cannot preserve the coloring, because any two vertices of distance two have distinct labels. $\square$

Next let us consider the ray and the double ray which can be viewed as an infinite analogons to finite paths. It turns out that their endomorphism distinguishing number is 2 as well.

Lemma 12 *The endomorphism distinguishing number of the infinite ray and of the infinite double ray is two.*

Later in this section Theorem 15 will show that every countable tree with at most one pendant vertex has endomorphism distinguishing number two. Clearly Lemma 12 constitutes a special case of this result. It is also worth noting that by the following theorem every double ray has infinite endomorphism motion. Hence we verify the Endomorphism Motion Conjecture for the class of countable trees.

Theorem 13 *A tree has infinite endomorphism motion if and only if it has no pendant vertices.*

The proof uses the following lemma which may be interesting on its own behalf. Note that in the statement of the lemma there is no restriction on the cardinality of the tree or the motion of the endomorphism.

Lemma 14 *Let T be a tree and let ϕ be an endomorphism of T. Then the set of fixed points of ϕ induces a connected subgraph of T.*

Proof. Denote by $\mathrm{Fix}(\phi)$ the set of fixed points of ϕ and assume that it does not induce a connected subgraph. Consider two vertices $v_1, v_2 \in \mathrm{Fix}(\phi)$ lying in different components of this graph.

Then ϕ maps the unique path in T from v_1 to v_2 to a v_1-v_2-walk of the same length. But the only such walk is the path connecting v_1 and v_2, so this path has to be fixed pointwise.

Proof of Theorem 13. Clearly, if a tree has a pendant vertex then there is an endomorphism which moves only this vertex and fixes everything else.

So let $T = (V, E)$ be a tree without pendant vertices and let ϕ be a nontrivial endomorphism of T. Assume that the motion of ϕ was finite. Then the set $\mathrm{Fix}(\phi)$ of fixed points of ϕ contains all but finitely many vertices of T. Since T has no pendant vertices such a set does not induce a connected subgraph. This contradicts Lemma 14. $\square$

Now that we have characterised the trees with infinite endomorphism motion, we would like to show that all of them have endomorphism distinguishing number 2. The following proof is due to F. Lehner.

Theorem 15 *The endomorphism distinguishing number of countable trees T with at most one pendant vertex is 2.*

Proof. The proof consists of two stages. First we color part of the vertices such that every endomorphism which preserves this partial coloring has to fix all distances from a given vertex v_0. Then we color the other vertices in order to break all remaining endomorphisms.

For the first part of the proof, let v_0 be a pendant vertex of T, or any vertex if T is a tree without pendant vertices. Denote by S_n the set of vertices at distance n from v_0, that is the sphere of radius n with center v_0. Now color v_0 white and all of S_1 and S_2 black. Periodically color all subsequent spheres according to the pattern outlined in Figure 2. In other words always color two spheres white, then four spheres black, one white, leave one sphere uncolored, color another four spheres black and proceed inductively. Furthermore, we require that vertices in the uncolored sphere S_k which have the same predecessor in S_{k-5} should be assigned the same color.

Now we claim that this coloring fixes v_0 in every endomorphism. To prove this consider a ray $v_0v_1v_2v_3\ldots$ starting at v_0. Clearly $v_i \in S_i$ holds for every i. Assume that there is a color preserving endomorphism ϕ of T which does not fix v_0 and consider the image of the previously chosen ray under ϕ, that is, let $\tilde{v}_i = \phi(v_i)$. Clearly $\tilde{v}_0$ has to lie either in a white sphere or in a sphere which has not yet been colored. We will look at those cases and show that all of them lead to a contradiction. So assume that $\tilde{v}_0 \in S_k$ for some $k > 0$.

- If $k = 3$, then $\tilde{v}_1$ must lie in S_2 since it must be a black neighbour of $\tilde{v}_0$. For similar reasons $\tilde{v}_2 \in S_1$ and $\tilde{v}_3 = v_0$ must hold. Now $\tilde{v}_4$ has to be a white neighbour of $\tilde{v}_3$ but v_0 only has black neighbors, a contradiction.

- If $k \in 3 + 12\mathbb{N}$ we get $\tilde{v}_1 \in S_{k-1}$ and $\tilde{v}_2 \in S_{k-2}$ by the same argument as above. Now $\tilde{v}_3$ would need to be a white neighbour of $\tilde{v}_2$ but $\tilde{v}_2$ only has black neighbors.

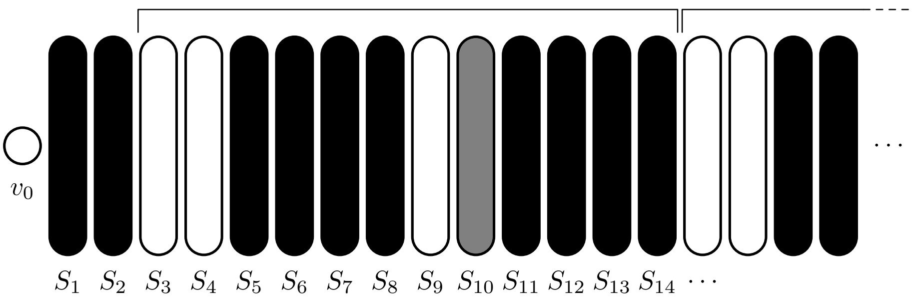

Figure 2: Coloring of the spheres in the first part of the proof of Theorem 15 with the period of the periodic part indicated at the top. Grey spheres are left uncolored for the second stage of the proof.

- If $k \in 4 + 12\mathbb{N}_0$ then, for similar reasons as in the previous cases, $\tilde{v}_1 \in S_{k+1}$ and $\tilde{v}_2 \in S_{k+2}$. Again $\tilde{v}_2$ has no white neighbors.

- If $k \in 9 + 12\mathbb{N}_0$ and $\tilde{v}_1 \in S_{k-1}$ we can argue precisely as in the previous cases. If $\tilde{v}_1 \in S_{k+1}$, then clearly $\tilde{v}_2 \in S_{k+2}$ must hold. Now $\tilde{v}_2$ has only black neighbors because its unique neighbour in S_{k+1} is $\tilde{v}_1$, which has to be colored black.

- Finally, if $k \in 10 + 12\mathbb{N}_0$, then $\tilde{v}_1 \in S_{k+1}$. Furthermore $\tilde{v}_2 \in S_{k+2}$, because the only neighbour of $\tilde{v}_1$ in S_k is $\tilde{v}_0$, which is colored white. Again all neighbors of $\tilde{v}_2$ are colored black.

Since there are no more cases left we can conclude that v_0 has to be fixed by every endomorphism which preserves this coloring.

However, we would like that every such endomorphism ϕ preserves all distances from v_0, that is, ϕ maps S_k into itself for each k. So assume that this is not the case and consider the smallest k such that $\phi(S_k) \nsubseteq S_k$. Then there must be some vertex $u_0 \in S_k$ such that $\phi(u_0) \in S_{k-2}$. This immediately implies that $k \notin \{1, 2\}$ and that $k \notin \{3, 4, 5, 6, 9, 11\} + 12\mathbb{N}_0$, because otherwise a white vertex would be mapped to a black vertex or vice versa. In order to treat the rest of the cases, consider a ray $u_0 u_1 u_2 u_3 \dots$ such that $u_i \in S_{k+i}$ and let $\tilde{u}_i = \phi(u_i)$.

- If $k \in 7+12\mathbb{N}_0$, then $\tilde{u}_1 \in S_{k-1}$ because S_{k-3} is colored white. But then $\tilde{u}_2$ has only black neighbors while $\tilde{u}_3$ has to be white.

- If $k \in 8 + 12\mathbb{N}_0$, then $\tilde{u}_0$ has only black neighbors, contradicting the fact that $\tilde{u}_1$ is white.

- If $k \in 10 + 12\mathbb{N}_0$, then it is immediate that u_0 must be black. Since $u_1, \dots, u_4$ are black as well, a parity argument shows that $\tilde{u}_4$ lies either in S_{k-2} or in S_{k-4}.

  In the latter case we immediadely get a contradiction because both S_{k-3} and S_{k-5} are colored black. In the first case clearly $\tilde{u}_5 \in S_{k-1}$. If we can show that u_0 and $\tilde{u}_5$

have the same predecessor in S_{k-5} then we are done, since in this case $\tilde{u}_5$ has only black neighbors.

But this is easy to see. The unique neighbour of u_0 in S_{k-1} is fixed, so we know that u_0 and $\tilde{u}_0$ have a common neighbour. Hence there is a path of length 7 connecting u_0 and $\tilde{u}_5$. This path certainly cannot pass through S_{k-5} and hence the two vertices have the same predecessor.

- If $k \in 12 + 12\mathbb{N}_0$, we can again use a parity argument to show that $\tilde{u}_2$ lies either in S_{k-2} or in S_k. In the first case – by a similar argument as above – $\tilde{u}_3 \in S_{k-3}$ has only black neighbors, while in the second case $\tilde{u}_2$ already has no white neighbour. Either way we can derive a contradiction.
- If $k \in 13 + 12\mathbb{N}_0$, then either $\tilde{u}_1 \in S_{k-1}$ and thus $\tilde{u}_1$ has no white neighbors or $\tilde{u}_1 \in S_{k-3}$. In this case by the same argument as before $w_2 \in S_{k-4}$ has no white neighbors.
- If $k \in 14 + 12\mathbb{N}_0$, then $\tilde{u}_0 \in S_{k-2}$ has no white neighbors.

This completes the proof of the fact that all distances from v_0 are fixed by any endomorphism which preserves such a coloring.

For the second part of the proof consider an enumeration $(v_i)_{i\geqslant 0}$ of all vertices such that every v_i is contained in S_j for some $j < 12i + 5$. Now color all vertices in S_{12i+10} whose predecessor is v_i black and color all other vertices in this sphere white.

We claim that the so obtained coloring is not fixed by any endomorphism but the identity. We already know that every color preserving endomorphism ϕ maps every sphere S_k into itself. Assume that there is a vertex v_i which is not fixed by ϕ. Then it is easy to see that all vertices in S_{12i+10} whose predecessor is v_i will be mapped to vertices whose predecessor is $\phi(v_i)$. Hence ϕ is not color preserving which completes the second part of the proof. □

Probably one can extend this result to uncountable trees. One does need a lower bound on the minimum degree though, see [10].

Furthermore, as we already noted, the fact that $D(T) = 2$, together with the observations that $|\mathrm{End}(T)| = \mathfrak{c}$ and $m_e(T) = \aleph_0$, supports the Endomorphism Motion Conjecture.

Of course, a proof of the Endomorphism Motion Conjecture is still not in sight, not even for countable structures.

Finally, the computation of $D_e(Q_k)$ seems to be an interesting problem, even for finite cubes. Similarly, the computation of $D_e(K_n^k)$, where K_n^k denotes the k-th Cartesian power[1] of K_n, looks demanding.

[1]For the definition of the Cartesian product and Cartesian powers see [8].

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