Distinguishing infinite graphs with bounded degrees

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Abstract

Call a colouring of a graph *distinguishing* if the only colour preserving automorphism is the identity. A conjecture of Tucker states that if every automorphism of a connected graph G moves infinitely many vertices, then there is a distinguishing 2-colouring. We confirm this conjecture for graphs with maximum degree $\Delta \leq 5$. Furthermore, using similar techniques we show that if an infinite graph has maximum degree $\Delta \geq 3$, then it admits a distinguishing colouring with $\Delta - 1$ colours. This bound is sharp.

1 Introduction

A distinguishing k-colouring of a graph $G = (V, E)$ is a map $c\colon V \to \{0, \dots, k-1\}$ such that the identity is the only automorphism γ with $c \circ \gamma = c$. The distinguishing number $D(G)$ is the least k such that G admits a distinguishing k-colouring. These notions were first introduced by Babai [1] in 1977 (under the name *asymmetric colouring* and *asymmetric number* respectively) and have since received considerable attention.

In this paper, we investigate connections between the maximum degree of a graph and its distinguishing number. This connection was studied in [3] and [8], where it was shown independently that connected finite graphs with maximum degree Δ satisfy $D(G) \leq \Delta + 1$ and that equality holds if and only if G is C_5, or K_r or $K_{r,r}$ for some $r \geq 1$. Imrich et al. [6] in 2007 proved that this remains true for infinite graphs, that is, $D(G) \leq \Delta$ for every connected infinite graph. Recently, Hüning et al. [4] completely characterized all (finite and infinite) connected graphs of maximum degree $\Delta \leq 3$ and distinguishing number 3, and Lehner and Verret [13] gave a similar characterization for finite, 4-regular, vertex transitive graphs. The results of [4] in particular show that all infinite connected graphs whose maximum degree is 3 satisfy $D(G) \leq 2$.

Our first main result is motivated by the following recent result of Babai [2] confirming a conjecture made by Tucker [15] over a decade ago.

Theorem 1. *If every non-trivial automorphism of a connected locally finite graph G moves infinitely many vertices, then G admits a distinguishing 2-colouring.*

Clearly, the requirement of connectedness cannot be omitted, Lehner and Möller [12] showed that the same is true for local finiteness.

It is worth pointing out that Babai's proof of Theorem 1 depends on what is perhaps the deepest result in group theory to date, the classification of finite simple groups (CFSG). The proof of the CSFG spans hundreds of research papers, see [14] for an overview. On the other hand, there are numerous graph classes for which Tucker's conjecture can be answered using purely combinatorial methods (see for example [4, 5, 6, 7, 9, 10, 11, 16]), thus raising the question whether Theorem 1 can be proved without the CFSG.

The paper [4] provides a combinatorial proof for graphs with maximum degree $\Delta \leq 3$. Our first main theorem is a further step towards a such a proof for graphs with bounded degrees.

Theorem 2. *Let G be a connected graph with maximum degree $\Delta \leq 5$ in which every non-trivial automorphism moves infinitely many vertices, then $D(G) \leq 2$.*

As indicated above, the proof is purely combinatorial, and we point out that most of the proof techniques generalize well to higher degrees. In fact, the only part of the proof that seems to break down for $\Delta > 5$ is Lemma 7. It is thus conceivable that a suitable generalization of this lemma would lead to a further improvement.

The result of Hüning et al. [4] shows that the bound obtained by Imrich et al. [6] in 2007 for the distinguishing number of connected infinite graphs with prescribed maximum degree is not tight in the case of subcubic graphs. Our second main result shows that this bound can be improved for every finite maximum degree greater than 2. It is easy to see that our bound is tight for every $\Delta \geq 3$. Note that the mentioned result of Hüning et al. [4] is a special case of the following theorem which constitutes the second main result of this paper.

Theorem 3. *Let G be an infinite connected graph with maximum degree $\Delta \geq 3$, then $D(G) \leq \Delta - 1$.*

2 Preliminaries

All graphs in this paper are simple, connected and locally finite. A multirooted graph (G, R) is a graph G together with a set $R \subseteq V$ of roots. Note that the set R is allowed to be empty or infinite; the latter possibility will play a role in the proofs of our main results. An automorphism of a multirooted graph (G, R) is an automorphism of G which fixes R pointwise. The group of automorphisms of (G, R) is denoted by $\mathrm{Aut}(G, R)$.

A partial colouring of a graph G is a function c from $S \subseteq V$ to a set C of colours. We denote the domain S of a partial colouring by $\mathrm{dom}(c)$. Call two partial colourings c and

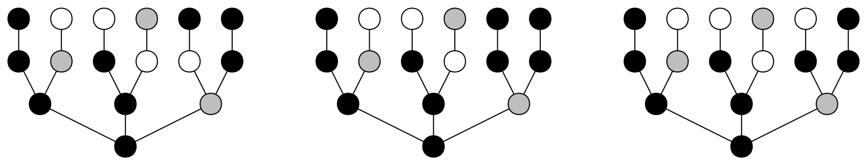

Figure 1: Three different partial colourings of a finite tree with colours black and white; uncoloured vertices are shown in gray. The first colouring is domain distinguishing, the second is domain preserving, and the third is neither. All three colourings are S-preserving for the set S of vertices in the central branch.

c' *compatible* if they coincide on $\operatorname{dom}(c) \cap \operatorname{dom}(c')$. For compatible colourings c and c' we define the partial colouring $c \cup c'$ with domain $\operatorname{dom}(c) \cup \operatorname{dom}(c')$ by

$$(c \cup c')(x) := \begin{cases} c(x) & x \in \operatorname{dom}(c), \\ c'(x) & x \in \operatorname{dom}(c'). \end{cases}$$

Note that this is well-defined as long as the colourings are compatible.

We say that a partial colouring c' *extends* a partial colouring c if $\operatorname{dom}(c) \subseteq \operatorname{dom}(c')$ and c' coincides with c on $\operatorname{dom}(c)$. Note that in this case $c \cup c' = c'$. Call a sequence $(c_i)_{i \in \mathbb{N}}$ of partial colourings *increasing* if c_{i+1} extends c_i for every $i \in \mathbb{N}$. For an increasing sequence of colourings define the limit colouring by

$$\lim_{i \to \infty} c_i := \bigcup_{i \in \mathbb{N}} c_i,$$

i.e. the colouring that maps every $v \in \operatorname{dom}(c_i)$ to $c_i(v)$.

We say that an automorphism $\gamma \in \operatorname{Aut}(G)$ *preserves* a partial colouring c if $c(v) = c(\gamma v)$ whenever both colours are defined, and call the set of all c-preserving automorphisms the *stabilizer* of c. Call a partial colouring c of a multirooted graph *S-distinguishing* if every c-preserving automorphism fixes S pointwise. Call it *S-preserving* if every c-preserving automorphism must fix S setwise. As a special case of this, we call a $\operatorname{dom}(c)$-distinguishing colouring *domain distinguishing*, and a $\operatorname{dom}(c)$-preserving colouring *domain preserving*; see Figure 1 for examples illustrating these definitions. Finally, a V-distinguishing colouring is simply called *distinguishing*. The *distinguishing number* of G is the least number of colours in a distinguishing colouring and denoted by $D(G)$.

The following two lemmas show how S-distinguishing colourings can be used to construct distinguishing limit colourings.

Lemma 4. *Let (G, R) be a multirooted graph. Assume that we have compatible colourings c and c', where c is an S-distinguishing colouring of (G, R), and c' is an S'-distinguishing colouring of $(G, R \cup S)$. Then $c \cup c'$ is an $(S \cup S')$-distinguishing colouring of (G, R).*

Proof. Let γ be a $(c \cup c')$-preserving automorphism. If γ moves a vertex in S, then this contradicts the fact that c was S-distinguishing. Hence, γ fixes $R \cup S$ which means that

γ is an automorphism of the multirooted graph $(G, R \cup S)$. Since c' is S'-distinguishing for $(G, R \cup S)$ this implies that γ cannot move any vertex in S', and hence $c \cup c'$ is $(S \cup S')$-distinguishing for (G, R). □

Lemma 5. *Let (G, R) be a multirooted graph. Let $(c_i)_{i \in \mathbb{N}}$ be an increasing sequence of partial colourings. Assume that c_i is S_i-distinguishing in (G, R) where the sets S_i satisfy $\bigcup_{i \in \mathbb{N}} S_i = V$. Then $\lim_{i \to \infty} c_i$ is distinguishing.*

Proof. Let γ be an automorphism that preserves $\lim_{i \to \infty} c_i$. Then γ preserves all partial colourings c_i. This implies that γ fixes S_i pointwise for every i, and since $V = \bigcup_{i \in \mathbb{N}} S_i$ we conclude that $\gamma = \mathrm{id}$. □

We say that a graph has *infinite motion* if every nontrivial automorphism moves infinitely many vertices. Theorem 1 states that a locally finite graph with infinite motion has distinguishing number at most 2. In the special case where the graph is a tree, this result boils down to the following theorem which has a purely combinatorial proof, see for instance [6, 16].

Theorem 6. *Let G be a locally finite tree with infinite motion, then $D(G) \leq 2$. In particular the distinguishing number of a locally finite tree without leaves is at most 2.*

3 Proof of the first main result

The proof strategy is to recursively construct a colouring and then use Lemma 5 to ensure that this colouring is distinguishing. Before giving any proof details, we briefly sketch an outline of the proof to motivate some auxiliary concepts.

The first step is to turn G into a multirooted graph by picking an infinite, connected root set R and assigning colour 0 to every vertex in R. Then recursively pick a vertex v and extend the current partial colouring in a way that every colour preserving automorphism fixes v. In case v is already coloured we have to achieve this by colouring some other vertices, which motivates the notion of *sychronization of moving tuples* below. At the end of this recursive construction, Lemma 5 ensures that no automorphism of (G, R) preserves the limit colouring c, but we still need to ensure that any automorphism in the stabilizer of c fixes R. To this end, we consider the subgraph H of G induced by the vertices with colour 0. The set R is contained in some infinite component of H. Our construction will imply that this component is asymmetric, so if it is fixed setwise, then is is fixed pointwise. This motivates the notion of *healthy and unhealthy components* below; intuitively, a healthy component is one that has some property which sets it apart from the component containing R. In other words, if all components of H are healthy, then any colour preserving automorphism fixes R setwise and thus also pointwise.

Before we give a formal proof of Theorem 2 we need formally introduce the notions from the proof sketch above and observe some basic facts about these notions.

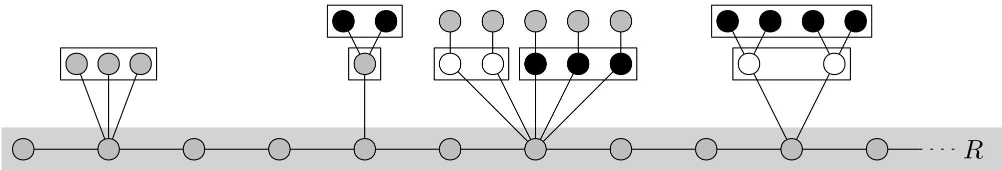

Figure 2: A partially coloured multirooted graph whose root set R is a ray; colours are black and white, uncoloured vertices are shown in gray. The boxed sets of non-root vertices form moving tuples. Note that the uncoloured vertices at distance 2 from R are not contained in moving tuples since they are uncharted.

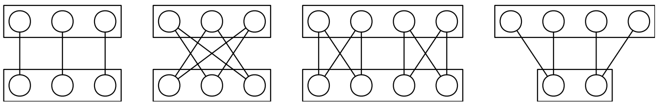

Figure 3: Moving tuples with uncommon neighbours. Note that the action on moving triples with uncommon neighbours in one another must be the same, but this is not true for moving quadruples.

3.1 Moving tuples and synchronization

Let c be a domain preserving colouring of a multirooted graph (G, R). Note that the stabilizer of a domain preserving colouring forms a group (this is not true in general for the stabilizer of a partial colouring). Call a vertex $v \in V \setminus R$ *charted* if it is coloured or it has a neighbour in R, and call v *uncharted* otherwise. A *moving tuple (singleton, pair, triple, quadruple,...)* is an orbit of a charted vertex under the stabilizer of c, see Figure 2. Since c is domain preserving, all vertices in a moving tuple must be charted.

Note that all members of a moving tuple must have the same neighbours in R. Outside of R there may be vertices that are adjacent to some, but not all members of a moving tuple. We call such vertices *uncommon neighbours* of the tuple. Uncommon neighbours can synchronize the action on different moving tuples as the following lemma demonstrates.

Lemma 7. *Let c be a domain preserving colouring with domain S, and let A and B be moving tuples with at most 3 elements. If A has an uncommon neighbour in B, then B has an uncommon neighbour in A and there is a bijection $f\colon A \to B$ such for every automorphism γ in the stabilizer of S we have $\gamma\mid_B = f \circ \gamma \mid_A \circ f^{-1}$. In particular, every such γ that fixes $v \in A$ must also fix $f(v) \in B$ and vice versa.*

Proof. First note that by the definition of moving tuples the stabilizer of S fixes A and B setwise but acts transitively on the elements of A and B respectively. Hence, every element of A has the same number of neighbours in B and vice versa. Since A and B both contain at most 3 vertices, this can only work if the graph induced by the edges between A and B is either complete bipartite, or a matching, or a 6-cycle. In the first case B cannot contain an uncommon neighbour of A. In the second case we take f to

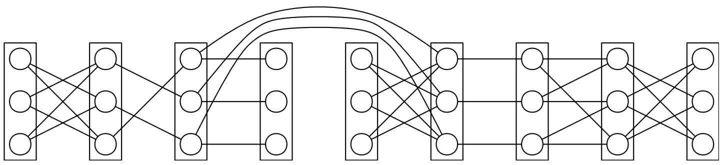

Figure 4: A set of synchronized moving triples. Note that by fixing a vertex in any triple we automatically fix a vertex in each of the triples.

be the map that takes each vertex to the other endpoint of its matching edge, in the last case let f be the map taking each vertex to the antipodal vertex on the cycle. □

Remark 8. Note that the lemma above is no longer true if we allow moving quadruples. In this case it is possible that we have two disjoint copies of $K_{2,2}$ between two moving quadruples or two disjoint copies of $K_{2,1}$ between a moving pair and a moving quadruple, see Figure 3. This is also why our proof does not easily carry over to higher maximum degrees.

We call two moving pairs or triples A and B synchronized and write $A \overset{\text{sync}}{\sim} B$ if there is a finite sequence $A = A_1, A_2, \ldots, A_k = B$ of moving pairs or triples such that A_i has an uncommon neighbour in A_{i+1} for $1 \leq i \leq k-1$, see Figure 4. By the lemma above, being synchronized is an equivalence relation. Furthermore, between any two synchronized moving pairs or triples we can find a bijection f such that an automorphism in the stabilizer of S fixes $x \in A$ if and only if it fixes $f(x) \in B$.

3.2 Healthy and unhealthy components

Let c be a partial colouring of a multirooted graph (G, R) and let v be a vertex in the domain of c. The *monochromatic component* of v is the set of all vertices that can be reached from v by a monochromatic path (i.e. no vertices of the opposite colour, but also no uncoloured vertices). Call a monochromatic component K *unhealthy*, if it satisfies the following properties:

- $K \cap R = \emptyset$,
- all vertices in K have colour 0,
- the maximum degree of the induced subgraph $G[K]$ is at most 3,
- either $|K| = \infty$, or K has an uncoloured neighbour.

If a monochromatic component is not unhealthy, we call it *healthy.* Call the colouring c *healthy* if all monochromatic components under c are healthy and *unhealthy* otherwise. A *symptom* of a colouring c is a vertex in an unhealthy component with uncoloured neighbours outside R, or an infinite unhealthy monochromatic component. Denote by $\mathrm{symp}(c)$ the set of symptoms of a colouring c. Note that a colouring c is healthy if and only if it has no symptoms.

Lemma 9. *Let c be a healthy colouring and let c' be a colouring extending c such that* $\mathrm{dom}(c') \setminus \mathrm{dom}(c)$ *is finite and contains no symptoms of c'. Then c' is healthy.*

Proof. If c' is unhealthy, then there is a symptom A of c'. By assumption, $A \cap \mathrm{dom}(c) \neq \emptyset$. If A is a vertex with uncoloured neighbours, then A is a symptom of c. Otherwise, there is an infinite monochromatic component of c contained in A, which is a symptom of c. In both cases we obtain a contradiction to the assumption that c is healthy. □

Lemma 10. *The limit colouring of any increasing sequence $(c_i)_{i\in\mathbb{N}}$ of healthy partial colourings is healthy.*

Proof. Assume that there is an unhealthy component K in the limit colouring. Let $i \in \mathbb{N}$ be such that $K_i := K \cap \mathrm{dom}(c_i) \neq \emptyset$. Since K is unhealthy in c, all vertices of K_i are coloured with colour 0, $K_i \cap R = \emptyset$, and no vertex of K_i has 4 or more neighbours in K_i. The colouring c_i is healthy, so K_i is finite and all neighbours of K_i except those in R have already been coloured (with colours different from 0). It follows that $K = K_i$, and thus K is healthy, a contradiction. □

3.3 Proof of Theorem 2

We are now ready to prove Theorem 2. Before doing so, let us briefly recall its statement.

Theorem 2. *Let G be a connected graph with maximum degree $\Delta \leq 5$ in which every non-trivial automorphism moves infinitely many vertices, then $D(G) \leq 2$.*

Proof. If G is a tree, then we are done by Theorem 6. We may thus assume that G contains at least one cycle. Let C be an induced cycle, let P be a geodesic ray which meets C only at its starting point, and let s be a neighbour of the starting point of P on C. Define $R = P + C - s$.

We now inductively define sets S_i with $\bigcup_{i\in\mathbb{N}} S_i = V$ and an increasing sequence c_i of partial colourings such that for every $i \in \mathbb{N}$

(C1) c_i is S_i-distinguishing in (G, R),

(C2) R is contained in a monochromatic component K_R of colour 0 such that

- each $r \in R$ has at most one neighbour in $K_R - R$, and
- each $v \in K_R - R$ has exactly one neighbour r in K_R which lies in $R - C$, neither v nor r have uncoloured neighbours,

(C3) c_i is healthy.

Before we construct these colourings, we show that their limit colouring is distinguishing.

First note that by Lemma 5 and (C1), the limit colouring $c = \lim_{i\to\infty} c_i$ is distinguishing for the multirooted graph (G, R). So we only need to show that every colour preserving automorphism fixes R pointwise.

It is easy to see that property (C2) carries over to the limit colouring. Hence, the monochromatic component K_R with respect to c is a ray with leaves attached to it. There is no leaf attached to the first two elements of the ray (because they are in C). Hence, any automorphism which fixes K_R setwise must fix R pointwise, and we only need to show that any colour preserving automorphism has to fix K_R setwise.

To this end, observe that by Lemma 10 and (C3), the limit colouring is healthy. In particular, K_R is the only infinite monochromatic component of colour 0 which does not contain a vertex of degree at least 4. Hence, K_R must be fixed setwise by every colour preserving automorphism.

In order to recursively construct the colourings c_i we make a few additional assertions which will be true for every $i \in \mathbb{N} \setminus \{0\}$.

(C4) c_i is domain preserving in (G, R),

(C5) $\mathrm{dom}(c_i) \setminus R$ is finite,

(C6) $\mathrm{dom}(c_i)$ contains all neighbours of $S_i \setminus R$,

(C7) there are no moving k-tuples for $k \geq 4$ in $\mathrm{dom}(c_i)$.

Let $S_1 = R$, and let c_1 be the colouring that assigns 0 to all vertices in R and leaves the remaining vertices uncoloured. Properties (C1) to (C6) are trivially satisfied. To see that (C7) also holds, recall that the vertices of R form a ray. Every internal vertex of this ray has two neighbours in R, and thus at most 3 neighbours outside R. The starting point of R may have 4 neighbours outside R, but at least one of them is also a neighbour of an internal vertex of R and hence contained in a moving k-tuple for $k \leq 3$.

Now assume that we already defined S_i and c_i. Let x be a vertex at minimal distance to C which is not contained in S_i. Note that x is a neighbour of S_i. Since by (C6) all neighbours of $S_i \setminus R$ are coloured, we conclude that x is charted. Let X be the moving tuple containing x.

We claim that if X is not a singleton, then one of the following holds:

1. There is an uncoloured $Y \overset{\text{sync}}{\sim} X$ whose unique coloured neighbour lies in $R - C$.

2. There is $Y \overset{\text{sync}}{\sim} X$ which has uncharted uncommon neighbours.

Let $\mathcal{Y}$ be the set of tuples that are synchronized with X. If no tuple in $\mathcal{Y}$ has an uncharted uncommon neighbour, then there is an automorphism that moves only vertices contained in $\mathcal{Y}$ and fixes all other vertices. Since G has infinite motion, this implies that $\mathcal{Y}$ must be infinite. By (C5), $\mathrm{dom}(c_i) \setminus R$ is finite, so infinitely many elements of $\mathcal{Y}$ must be neighbours of R.

Note that every internal vertex of R has neighbours in at most one moving tuple in $\mathcal{Y}$, otherwise, its degree would be larger than 5 since it already has two neighbours in R. It's easy to see that there are tuples $Y_1, Y_2, Y_3 \in \mathcal{Y}$ whose only coloured neighbours lie on the geodesic ray $P \subseteq R$ such that Y_i has uncommon neighbours in Y_{i+1} for $i \in \{1, 2\}$. Since each Y_i has 2 or more neighbours in R, we conclude that at least two of those neighbours, v and w, lie at distance 5 or more from each other on R.

Assume that v is a neighbour of Y_1 and w is a neighbour of Y_3. Then there is a path v, y_1, y_2, y_3, w, where $y_{i+1} \in Y_{i+1}$ is an uncommon neighbour of $y_i \in Y_i$ for $i \in \{1, 2\}$. This path has length 4, contradicting the fact that P was geodesic. The cases where v and w are neighbours of other tuples in Y_1, Y_2, Y_3 are completely analogous and lead to even shorter paths. This finishes the proof of the claim.

Let $S_{i+1} = S_i \cup \{x\}$. We now define the colouring c_{i+1} by first colouring $\mathrm{dom}(c_i)$ according to c_i, and then possibly colouring some yet uncoloured vertices. If X is uncoloured (this can only happen if x is a neighbour of R), then colour all elements of X with 1. Note that if $X = \{x\}$ is a moving singleton with no uncoloured neighbours, then (C1) to (C7) are trivially satisfied, whence we found c_{i+1}.

Otherwise, distinguish cases according to the two possible options arising from the claim above. The case where $X = \{x\}$ is a singleton with uncoloured neighbours will be treated together with the case where there are uncharted uncommon neighbours of some $Y \overset{\text{sync}}{\sim} X$.

First assume that there is an uncoloured $Y \overset{\text{sync}}{\sim} X$ whose unique coloured neighbour r lies in $R - C$. By Lemma 7 there is an element $y \in Y$ such that any c_i-preserving automorphism that fixes y must also fix x. Colour y with 0 and all other uncoloured neighbours of r with 1. Colour all uncoloured neighbours of x and y with colour 1 and let c_{i+1} be the resulting colouring.

It remains to show that c_{i+1} satisfies (C1) to (C7). For (C1) note that any c_{i+1}-preserving automorphism also preserves c_i and thus fixes S_i pointwise and Y setwise. Since y is the only vertex with colour 0 in Y, every c_{i+1}-preserving automorphism must fix y and thus also x. If r had any neighbours of colour 0 in c_i, then c_i would violate (C2). Hence, the way we picked y and the fact that we coloured all remaining neighbours of y and r with 1 ensures that (C2) holds for c_{i+1}. Property (C3) holds by Lemma 9 and the fact that y is the only vertex with colour 0 in $\mathrm{dom}(c_{i+1}) \setminus \mathrm{dom}(c_i)$. For (C4) note that c_i was domain preserving, hence every vertex in $\mathrm{dom}(c_i)$ is mapped to a coloured vertex. Furthermore, every c_{i+1}-preserving automorphism must fix x, y, and r and hence fixes their neighbourhoods setwise. Properties (C5) and (C6) hold because we coloured finitely many vertices including the neighbourhood of x. For the proof of (C7) first note that when we picked Y, the tuple X was already coloured. So $X \neq Y$ and consequently both x and y have at least two c_i-charted neighbours: one in a synchronized moving tuple, and one in S_i. All other vertices in $\mathrm{dom}(c_{i+1}) \setminus \mathrm{dom}(c_i)$ are c_i charted. Since c_i satisfies (C7), all charted vertices are contained in moving k-tuples for $k \leq 3$, and we conclude that c_{i+1} satisfies (C7) as well.

Finally, consider the case when there is $Y \overset{\text{sync}}{\sim} X$ which has uncharted uncommon neighbours. If X has uncharted uncommon neighbours, we pick $Y = X$. Let $y \in Y$ be such that every c_i-preserving automorphism that fixes y must also fix x. The argument below also applies in the case where X is a singleton with uncoloured neighbours. In this case let $Y = X$ and $y = x$.

If $X \neq Y$ and X has uncoloured neighbours, then colour all of them with colour 1. If Y is uncoloured, then colour it with colour 1. Let $\mathcal{A} = Y$ and iteratively run the following procedure.

Let $a \in \mathcal{A}$ (in the first step choose $a = y$) and let A be the moving tuple containing a. Colour all neighbours of A that are also neighbours of R with colour 1. If there are still uncoloured neighbours, we distinguish the following cases.

1. A has no uncoloured uncommon neighbours (this includes the case where $|A| = 1$).
   - 1A If A has 3 or fewer uncoloured neighbours, do nothing.
   - 1B If A has 4 uncoloured neighbours, then colour 3 of them with 0.
2. $A = \{a_1, a_2\}$ is a moving pair, or $A = \{a_1, a_2, a_3\}$ is a moving triple with uncoloured uncommon neighbours. Without loss of generality, $a_1 = a$. We say that an uncommon neighbour v of A is *linked to* a_i if either a_i is the unique neighbour of v in A, or if A is a moving triple and a_i is the unique vertex in A that is not connected to v.
   - 2A If there is a unique uncommon neighbour linked to a_1, then colour it with 0.
   - 2B If there are 2 or 3 uncommon neighbours linked to a_i, then colour $(i - 1)$ uncommon neighbours linked to a_i with colour 0.
   - 2C If there are 4 uncommon neighbours linked to a_i, then colour $(4-i)$ uncommon neighbours linked to a_i with colour 0.

Finally, colour all remaining uncoloured neighbours of A with colour 1. If any of the newly coloured vertices are symptoms, then add them to $\mathcal{A}$. Remove all elements from $\mathcal{A}$ that have ceased to be symptoms. If $\mathcal{A} \neq \emptyset$, iterate.

We claim that after finitely many iterations this yields the desired colouring c_{i+1}. We first show that (C1), (C2), and (C4) to (C7) hold after every step of the recursion. Then we prove that the recursion eventually terminates with $\mathcal{A} = \emptyset$ and that this implies that the colouring is healthy.

We show inductively that all desired properties except (C3) hold after every iteration. The induction basis for (C2), (C4), and (C5) trivially follows from the corresponding properties of c_i.

For the induction basis for (C1) first note that if $|X| = 1$, then this is trivially satisfied (even before the first iteration). To see that if $|X| > 1$ it is satisfied after the first iteration, note that Y has uncharted uncommon neighbours. Hence, it still has uncoloured uncommon neighbours after colouring the neighbours of R in its neighbourhood, i.e. we apply Case 2. In each of the subcases it is easy to see that the number of uncommon neighbours of $y = a_1$ with colour 0 differs from the corresponding numbers for a_2 and a_3, whence y (and thus also x) is fixed by every colour preserving automorphism.

If $Y = X$, then property (C6) will be satisfied after the first iteration where all neighbours of Y are coloured, otherwise it follows from the colouring that was done before the first iteration.

Property (C7) always holds before the first iteration. If $X = Y$, then this follows from (C7) for c_i. Otherwise, X has no uncharted uncommon neighbours, so all uncharted neighbours of X are neighbours of x. Note that x has at least one neighbour in S_i and one neighbour in a synchronized moving tuple, and thus X has at most 3 uncharted neighbours. Since c_i satisfies (C7), every charted neighbour of X is contained in moving k-tuples for some $k \leq 3$. Hence, there are no moving k-tuples for $k \geq 4$ in the neighbourhood of X and it follows that the colouring before the first iteration satisfies (C7).

For the induction step note that the colouring procedure preserves property (C1) since an extension of an S-distinguishing colouring is again S-distinguishing. It preserves (C2) since throughout the procedure neighbours of root vertices are only coloured with colour 1. Property (C4) is preserved since the whole neighbourhood of an orbit is coloured in each recursion step. The colouring procedure preserves (C5) since only finitely many vertices are coloured in each iteration. It also trivially preserves (C6).

For the induction step for (C7) note that in all cases except 2A with a moving triple, every element of A has a different number of neighbours with colour 0. In particular, A must be fixed pointwise by every colour preserving automorphism and the colouring on the neighbourhood of A makes sure that there are no moving k-tuples for $k \geq 4$. If A is a moving triple and we are in case 2A, then a_1 is fixed by every colour preserving automorphism, and so is the unique previously uncoloured neighbour that is linked to a_1. The vertices a_2 and a_3 are either fixed by every colour preserving automorphism or they form a moving pair, and the same is true for the previously uncoloured vertices that are linked to them. Every remaining uncoloured neighbour of A is not uncommon and hence connected to every vertex of A. Since a_1 had at least one coloured vertex before the iteration (otherwise it would not have been added to $\mathcal{A}$) we conclude that there are at most 3 uncoloured neighbours of $\mathcal{A}$ left, and hence there is no moving k-tuple for $k \geq 4$.

Next we show that the recursion terminates. For this purpose first note that after each iteration all neighbours of every vertex in A are coloured. Hence, the members of A are removed from $\mathcal{A}$ and won't be added again in later iterations. Hence, it suffices to show that the number of vertices that can possibly be added to $\mathcal{A}$ is finite.

To this end define the generation $g(x)$ of an element $x \in \mathcal{A}$ as follows. The generation of any element of Y is 1. If x has been added to $\mathcal{A}$ in an iteration where $g(a) = i$, then $g(x) = i + 1$. Note that every vertex in generation $i + 1$ has at least one neighbour in generation i and thus lies at distance at most $i - 1$ from Y. So it suffices to show that there is an upper bound on $g(x)$.

Every vertex added to $\mathcal{A}$ is a symptom and thus has colour 0. Hence, every vertex of generation 3 and higher has a neighbour of colour 0 of an earlier generation. Note that moving triples of colour 0 can only be created in Subcases 1B and 2C. In case a already has a neighbour of colour 0, hence the degree of a in its monochromatic component increases to 4 thus making this monochromatic component healthy. In particular, such triples will not be added to $\mathcal{A}$ since they are contained in a healthy component. It follows that no $x \in \mathcal{A}$ with $g(x) \geq 4$ is contained in a moving triple.

Next note that if A is a moving pair, then A is fixed pointwise by every automorphism that preserves the colouring after the iteration. Hence, the vertices of any moving tuple that is added to $\mathcal{A}$ in this iteration must have a common neighbour in A. In particular, the members of a moving tuple of generation 5 or higher lie in the same monochromatic component. If Subcase 2C is applied to such a tuple, no vertices are added to $\mathcal{A}$. Since all other cases only yield moving singletons of colour 0, we conclude that every $x \in \mathcal{A}$ with $g(x) \geq 6$ is fixed pointwise by any colour preserving automorphism. Neither of the subcases of Case 1 yields new elements of $\mathcal{A}$ after generation 4 and it follows that there are no vertices of generation 7.

Finally, we need to show that the colouring c_{i+1} obtained by running the recursion until $\mathcal{A} = \emptyset$ is healthy. Assume that there is a symptom $v \in \operatorname{dom}(c_{i+1}) \setminus \operatorname{dom}(c_i)$. Then v would have been a symptom after the iteration in which it was coloured, and hence we would have added v to $\mathcal{A}$. Since we ran the recursion until $\mathcal{A} = \emptyset$, there must have been an iteration in which v was removed from $\mathcal{A}$ due to not being a symptom any more. This implies that v is not a symptom of any subsequent colouring and in particular v is not a symptom of c_{i+1}. Since there are no symptoms in $\operatorname{dom}(c_{i+1}) \setminus \operatorname{dom}(c_i)$, we can use Lemma 9 and conclude that c_{i+1} is healthy. □

It is obvious that by using more colours in the colouring procedure above we can avoid moving k-tuples for $k \geq 4$ even if the maximum degree is larger than 5. Consequently, the proof can be adapted to show the following.

Corollary 11. *Let G be a connected graph with maximum degree $\Delta \geq 3$ in which every non-trivial automorphism moves infinitely many vertices. Then $D(G) \leq \frac{\Delta}{3} + 1$.*

4 A general bound

In light of the corollary above it is only natural to ask how much the bound for $D(G)$ changes if we allow automorphisms moving only finitely many vertices. Recall that by [6] we know that $D(G) \leq \Delta$ for any infinite graph with maximum degree Δ. On the other hand, it is not hard to construct examples of infinite graphs with $D(G) = \Delta - 1$. Simply attach $\Delta - 1$ new leaves to a vertex of degree 1 in any infinite graph. Hence, the bound given in Theorem 3 (which we recall below) is tight.

Theorem 3. *Let G be an infinite connected graph with maximum degree $\Delta \geq 3$, then $D(G) \leq \Delta - 1$.*

Proof. If G is a leafless tree, then we are done by Theorem 6.

Hence, we may assume that G has either a cycle or a vertex of degree 1. If there is a cycle, then let C be an induced cycle, P be a geodesic ray which meets C only at its starting point, and let s be a neighbour of the starting point of P on C. Define $R = P + C - s$. If G is a tree with leaves, then let R be a ray starting at a leaf. In both cases define r as an endvertex of R.

We will now inductively define domain distinguishing partial colourings c_i of (G, R) with connected domain $S_i = \operatorname{dom}(c_i)$. For c_0 colour all vertices of R with colour 0. This is trivially domain distinguishing for (G, R) and clearly $S_0 = R$ is connected.

For the recursion step define an equivalence relation on $V \setminus S_i$ by $x \sim y$ if x and y have the same neighbours in S_i. Denote by $[x]$ the equivalence class of x with respect to this relation. Let $v \notin S_i$ be an uncoloured vertex which lies as close as possible to r. Then v has a neighbour s in S_i. Since S_i is connected, s has at most $\Delta - 1$ neighbours outside

S_i whence $|[v]| \leq \Delta - 1$. Colour all vertices in $[v]$ with different colours. If $|[v]| \leq \Delta - 2$, avoid the colour 0. The resulting colouring c_{i+1} clearly is domain distinguishing for (G, S_i). By Lemma 4 and the induction hypothesis it is also domain distinguishing for (G, R).

The sequence c_i satisfies the conditions of Lemma 5, thus we get a limit colouring c which is distinguishing for (G, R). In order to show that c is distinguishing it suffices to show that every colour preserving automorphism must fix R pointwise.

This can be achieved by showing that R is the unique monochromatic ray of colour 0 satisfying the following property:

($*$) the first vertex is either a leaf, or has a common neighbour outside R with another vertex of R.

Clearly R satisfies ($*$). If v is a neighbour of R, then $|[v]| \leq \Delta - 2$. This is obvious, if v is a neighbour of an inner vertex of the ray induced by R. For the starting vertex it follows from the fact that this vertex either has degree 1 or has a common neighbour with an inner vertex of the ray. Hence, no neighbour of R is coloured with colour 0 showing that any other ray satisfying ($*$) must be disjoint from R.

Further, observe that if u and v are neighbours which are both coloured 0 then they must have been coloured in different steps of the construction. If v is coloured later than u, then $\Delta - 1$ neighbours of u (including v) are coloured in the same step as v.

Now let $Q = q_0 q_1 q_2 \dots$ be a monochromatic ray of colour 0 disjoint from R. We claim that the vertices of Q are coloured in order, i.e. q_i is coloured before q_{i+1}. Indeed, if q_{i+1} is coloured earlier than q_i, then let q be the first vertex of $\{q_j \mid j \geq i\}$ that is coloured. Both neighbours of q on Q are coloured later than q. Hence, by the observation above vertex q has at least $2(\Delta - 1)$ neighbours, a contradiction to Δ being the maximum degree.

At least one neighbour of q_0 is coloured before q_0, so if q_0 is a leaf, then q_1 can't be coloured after q_0. Furthermore, once q_1 is coloured all neighbours of q_0 are coloured. But for $i > 0$ the only neighbour of q_i which is not coloured later than q_i is q_{i-1}, whence q_i can't have any common neighbours with q_0 outside Q. Thus, Q does not satisfy ($*$). □

5 Outlook and open questions

As already mentioned, the recent proof of Tucker's infinite motion conjecture given by Babai [2] depends on the classification of finite simple groups. An obvious question is, whether this dependency can be avoided; the results of the present paper suggest that this may be easier if we bound the maximum degree.

Question 12. *Is there a combinatorial proof that graphs with maximum degree* Δ *and infinite motion have distinguishing number at most* 2*?*

While Theorem 1 establishes a link between the motion and the distinguishing number of a graph, it is likely that this connection is a lot stronger for graphs with bounded maximum degree. The following question was posed by Babai at the BIRS/CMO workshop on 'Symmetry Breaking in Discrete Structures' in September 2018.

Question 13. *Is there a function f such that any graph with maximum degree Δ whose motion is larger than $f(\Delta)$ admits a distinguishing colouring with 2 colours?*

Results from [4] show that $f(3) = 2$, and it follows from [13] that $f(4)$ exists if we restrict ourselves to finite, vertex transitive graphs. While the present paper shows that graphs with maximum degree 4 and 5 can be treated combinatorially, our methods are not strong enough to answer the above question for $\Delta = 4$ or $\Delta = 5$.

Problem 14. *Find $f(4)$ and $f(5)$ as defined in the above question, or show that the values do not exist.*

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