On the cop number of toroidal graphs

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Abstract

We show that the cop number of toroidal graphs is at most 3. This resolves a conjecture by Schroeder from 2001 which is implicit in a question by Andreae from 1986.

1 Introduction

Cops and Robber is a pursuit–evasion game played on a graph between two players. Originally introduced independently by Nowakowski and Winkler [9], and Quilliot [10], this game and variants thereof have become a quickly growing research area within graph theory. The book [5] provides an extensive introduction to the topic.

The variant considered in this paper was first studied by Aigner and Fromme [1] and can be described as follows. Initially, the first player, called Cops, places k cops[1] on the vertices of a graph G. Then the second player, called Robber, places a robber on a vertex. Then the two players take turns. On Cops' turn, each cop can either be moved to an adjacent vertex or left at the current position. On Robber's turn, the robber can either be moved to an adjacent vertex or left where he is. Both players have perfect information, that is, they know the other player's moves and possible strategies. Cops wins the game if at some point one of the cops is at the same vertex as the robber, in this case we say that the robber is caught.

One of the most studied questions concerning this game is whether for some given k there is a winning strategy for Cops using k cops. The *cop number* of a graph G, denoted by $c(G)$, is the least positive integer k for which Cops has a winning strategy. The most famous open problem in this context is Meyniel's conjecture, stating that the cop number of any graph on n vertices is at most $O(\sqrt{n})$. If true, this is asymptotically tight since there are graph classes meeting this bound. However, not even an upper bound of the form $O(n^{1-\epsilon})$ is known, see [3] for an overview.

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[1]Throughout this note, we use Cops to refer to the player, and cops to refer to the playing pieces of that player on the graph. An analogous distinction is made between Robber and robber.

Bounds for the cop number have also been studied in certain graph classes, with an increased recent interest in graph classes defined by topological invariants, see for example the survey [4]. Andreae [2] showed that for any fixed graph H there is a constant upper bound on the cop number of connected graphs with no H-minor. It follows that there is a constant upper bound on the cop number of connected graphs of genus g. In his paper, Andreae also poses the question of finding sharp bounds for the cop number of such graphs in terms of g.

So far, such a bound is only known for $g = 0$. Aigner and Fromme [1] showed that on any connected planar graph Cops has a winning strategy using 3 cops, and there are planar graphs (such as the dodecahedron) for which 3 cops are necessary. For toroidal graphs G, Quilliot [11] proved an upper bound of $c(G) \leq 5$, and Andreae [2] asked whether this could be improved to $c(G) \leq 3$. Schroeder [12] improved Quilliots bound to $c(G) \leq 4$, and explicitly stated the conjecture implicit in Andreae's question.

Conjecture 1.1 (Andreae [2], Schroeder [12])**.** *Let G be a finite toroidal graph, then $c(G) \leq 3$.*

In this short note we prove this conjecture. This is done by relating Cops and Robber on a graph G to a similar game with more powerful Cops (which we call T-Cops and Robber) on a cover of G. We note that similar ideas have been used in [7], but without increasing the Cops' power which is crucial for our proof to work.

As a corollary to our main result, we are able to make progress on the following conjecture of Schroeder [12].

Conjecture 1.2 (Schroeder [12])**.** *Let G be a finite graph of genus g, then $c(G) \leq g + 3$.*

The best known general bound is $c(G) \leq \frac{4}{3}g + \frac{10}{3}$, proved in [6], but so far the conjecture is only known to hold for $g \leq 2$. We give a simpler proof for the case $g = 2$, and prove the case $g = 3$.

While this confirms Conjecture 1.2 for $g \leq 3$, the bound is only known to be tight for $g = 0$. Tightness fails for $g = 1$ by our main result, thus raising the following question.

Question 1.3. *Is there any graph with genus $g > 0$ and cop number equal to $g + 3$?*

Maybe even more fundamentally, we do not know whether the bound in Conjecture 1.2 is asymptotically tight (Mohar in [8] conjectured that it is not), which shows how little is known about the interplay between the genus and the cop number of a graph. In fact, to our best knowledge even the following question is still open.

Question 1.4. *What is the smallest g such that there is a graph with genus g and cop number 4?*

2 Preliminaries

Throughout this paper, let $G = (V, E)$ be a graph. All graphs considered are simple, undirected, and locally finite (every vertex only has finitely many neighbours). As

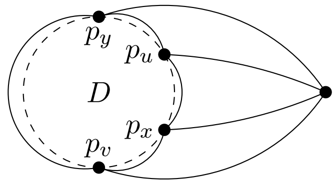


Figure 1: Embedding of a graph; embeddings of edges meet the disk D only in their endpoints.

usual, let d denote the geodesic distance on V, that is, $d(u, v)$ is the length of a shortest path from u to v. For $v \in V$ and $r \in \mathbb{N}$ define the *ball around v with radius r* by $B(v, r) = \{w \in V \mid d(v, w) \leq r\}$. The ball with radius 1 around v is called the *closed neighbourhood of v* and denoted by $N[v]$. The *growth function* of G around v is the function $g \colon \mathbb{N} \to \mathbb{N}$ defined by $g(n) = |B(v, n)|$. We say that a graph has *polynomial growth*, if the growth function around some (or equivalently any) of its vertices is upper bounded by a polynomial.

An *embedding* of G on a surface S assigns to each vertex v a point p_v on S and to each edge $e = uv$ an arc a_e connecting p_u to p_v such that

1. the points $(p_v)_{v \in V}$ are distinct,
2. the arcs $(a_e)_{e \in E}$ are internally disjoint, and
3. no point p_v lies in the interior of an arc a_e.

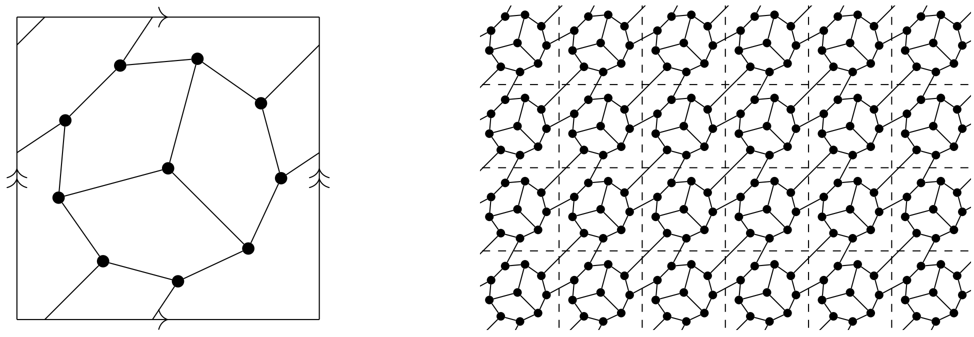
Clearly, given a set of points and arcs on a surface with the above properties we can find a graph with this embedding. Call an embedding *accumulation free*, if the set $\{p_v \mid v \in V\}$ has no accumulation points. A graph is called *planar* if it has an embedding in the plane $\mathbb{R}^2$ and *toroidal* if it has an embedding in the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The following observation about embeddings will be useful.

Lemma 2.1. *Let G be a finite graph embedded on a closed disk D, and let u, v, x, y be four vertices whose embeddings p_u, p_v, p_x, p_y lie on the boundary ∂D of D. If p_x and p_y lie in different connected components of $\partial D \setminus \{p_u, p_v\}$, then every path connecting u to v meets every path connecting x to y .*

Proof sketch. If there were disjoint paths connecting u to v and x to y, then their embeddings would give rise to disjoint arcs in D connecting p_x to p_y and p_u to p_v respectively. These two arcs together with the embedding shown in Figure 1 would give a planar embedding of K_5 contradicting the well known fact that K_5 is not planar. □

A graph $\hat{G} = (\hat{V}, \hat{E})$ is a *cover* of G, if there is a surjective map $\phi : \hat{V} \to V$ such that ϕ is a bijection from $N[v]$ to $N[\phi(v)]$ for every $v \in \hat{V}$. The map ϕ is called a *covering map*.

The following result is almost trivial and probably known, but we couldn't find a reference for it in the literature which is why we provide a proof sketch for the convenience of the reader.



(a) (b)

Figure 2: (a) An embedding of the Petersen graph in the torus and (b) part of the corresponding planar cover constructed in Lemma 2.2. Note that the drawing in each of the dashed squares on the right is exactly the same as the drawing on the left.

Lemma 2.2. *If G be a finite toroidal graph, then there is an infinite planar cover $\hat{G}$ of G with polynomial growth. Moreover, $\hat{G}$ has an accumulation free embedding in $\mathbb{R}_2$.*

Proof sketch. Let G be a toroidal graph and let $(p_v)_{v\in V}, (a_e)_{e\in E}$ be an embedding of G in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ be the usual projection map, that is, $\pi(x) = x + \mathbb{Z}^2$. For $v \in V$ define the set $P_v = \pi^{-1}(p_v)$, and for $e \in E$ let A_e be the set of connected components of $\pi^{-1}(a_e)$. In other words, P_v is the set of all points in $\mathbb{R}^2$ that project to the embedding p_v of v in $\mathbb{T}^2$, and A_e is a collection of arcs in $\mathbb{R}^2$ each of which projects to the embedding a_e of e in $\mathbb{T}^2$, see Figure 2 for an example. It is readily verified that the set of points $P = \bigcup_{v\in V} P_v$ together with the set of arcs $A = \bigcup_{e\in E} A_e$ defines an accumulation free embedding of a graph $\hat{G} = (\hat{V}, \hat{E})$ in the plane and that the projection π gives rise to a covering map by mapping $\hat{v}$ to v if $\pi(p_{\hat{v}}) = p_v$.

To show polynomial growth, note that in the embedding of $\hat{G}$ defined above, exactly $|V|$ vertices embed into any translate of $[0, 1)^2$. Since any two arcs in A_e can be mapped into each other by a translation, there is an absolute upper bound $R \in \mathbb{N}$ on the Euclidian distance of the embeddings of two neighbours in $\hat{G}$. Consequently, the embeddings of all vertices in $B(v, r)$ are contained in some translate of $[-rR, rR + 1)^2$ and thus $B(v, r)$ contains at most $(2rR + 1)^2 \cdot |V|$ vertices. □

The Cops and Robber game on G with k cops is a game played on G between two players, who are called Cops and Robber respectively. In the beginning of the game, Cops picks $(c_0^1, c_0^2, \ldots, c_0^k) \in V^k$, then Robber picks $r_0 \in V$. In each subsequent turn n, Cops picks $c_n^i \in N[c_{n-1}^i]$, then Robber picks $r_n \in N[r_{n-1}]$. Cops wins the game, if $c_{n+1}^i = r_n$ or $c_n^i = r_n$ for some $n \in \mathbb{N}$ and some $1 \leq i \leq k$. Note that an optimally playing Robber can make sure that the latter option does not happen first, whence we could also insist on $c_{n+1}^i = r_n$ as a winning criterion. The *cop number* $c(G)$ is the least k such that Cops has a winning strategy.

Intuitively, we think of the c_n^i and r_n as the position of playing pieces on the graph, COPS' playing pieces are thought of as k cops, ROBBER's piece is thought of as a robber. Using this intuition, the winning criterion for COPS says that some cop catches the robber by moving to the same vertex. We say that a subgraph H of G is *i-guarded at time n*, if $r_n \in H$ implies that $c_{n+1}^i = r_n$, independently of the strategy used for the other cops. Intuitively this means that COPS is using the i-th cop to make sure that the robber cannot move to H without being caught. Call a subgraph H *guarded*, if it is i-guarded for some $i \leq k$.

3 Proof of the main result

Given an equivalence relation T on V we can define the following variant of COPS AND ROBBER, which we call T-COPS AND ROBBER. The rules are the same as in the original game, except COPS is able to 'teleport cops to an equivalent position' before moving them. More formally, she can pick $\tilde{c}_{n-1}^i \, T \, c_{n-1}^i$ and choose $c_n^i \in N[\tilde{c}_{n-1}^i]$. The *$T$-cop number* $c^T(G)$ is the least k such that COPS has a winning strategy using k cops in T-COPS AND ROBBER. Note that we do not allow ROBBER to teleport—this is essential for Lemma 3.3 which otherwise would have to be replaced by an even more technical statement.

While the above game is defined for any equivalence relation T, we are only interested in the special case given by the following definition. It is worth pointing out that we will in fact only need the (even more special) case where G is toroidal and $\hat{G}$ is the cover from Lemma 2.2.

Definition 3.1. For the remainder of this section, we fix a finite graph $G = (V, E)$, a (not necessarily finite) cover $\hat{G} = (\hat{V}, \hat{E})$ of G, and a covering map $\phi\colon \hat{G} \to G$. Moreover, throughout the remainder of this section T will always denote the equivalence relation defined by $v \, T \, w$ if and only if $\phi(v) = \phi(w)$.

The following lemma connecting COPS AND ROBBER on G to T-COPS AND ROBBER on $\hat{G}$ is very similar to [7, Lemma 1]. We remark that the converse inequality also holds, but we will not need this fact.

Lemma 3.2. $c(G) \leq c^T(\hat{G})$.

Proof. Assume that COPS has a winning strategy for T-COPS AND ROBBER with k cops on $\hat{G}$. We define a strategy of COPS on G by projecting such a winning strategy onto G.

More precisely, given the initial position r_0 chosen by ROBBER on G, we pick $\hat{r}_0$ arbitrarily with $\phi(\hat{r}_0) = r_0$. For $n \geq 1$, let r_n be the position of ROBBER on G at time n, and inductively pick $\hat{r}_n \in N[\hat{r}_{n-1}]$ such that $\phi(\hat{r}_n) = r_n$. Note that $\hat{r}_n$ is unique since ϕ is a covering map. The strategy for COPS on G is given by $(\phi(\hat{c}_n^1), \ldots, \phi(\hat{c}_n^k))$ where $(\hat{c}_n^1, \ldots, \hat{c}_n^k)$ is the position of COPS on $\hat{G}$ with respect to the winning strategy played against $\hat{r}_n$. Note that this is a valid strategy because ϕ is a covering map.

Since the strategy of Cops on $\hat{G}$ is winning, there is some n and i such that $\hat{c}_n^i = \hat{r}_{n-1}$. This clearly implies $c_n^i = \phi(\hat{c}_n^i) = \phi(\hat{r}_{n-1}) = r_n$, so Cops wins the game on G in the same move or earlier. □

A weaker version of the next lemma can be found in [1]. The advantage of our version is that we can use the additional power of Cops in T-Cops and Robber to obtain a bound the distance between u and r_j until the path P is guarded. This will be essential in the proof of our main result.

Lemma 3.3. *Let $u, v \in \hat{V}$, and let let P be a shortest u-v-path. Let r_n, c_n^i be positions in T-Cops and Robber on $\hat{G}$ with k cops at time n, and let $i_0 \in \{1, \dots, k\}$. Then there is a strategy for* Cops *such that for some $m > n$ the following hold:*

1. *$d(u, r_j) \leq d(u, r_n) + |V|$ for $n \leq j \leq m$,*
2. *P is i_0-guarded at all times $j \geq m$.*

Furthermore this strategy does not depend on how c_j^i evolve for $i \neq i_0$ (the value of m, however, depends on Robber*'s strategy which in turn may depend on all c_j^i).*

Proof. Without loss of generality take $i_0 = 1$ and $n = 0$. We give a strategy with the desired properties.

Let D be the length of P and let x be the unique vertex on P satisfying $d(u, x) = \min(D, d(u, r_0) + |V|)$—uniqueness follows from the fact that P is a shortest path. Since ϕ is a covering map, it can be used to lift any path from $\phi(x)$ to $\phi(c_0^1)$ in G to a path from x to some $\tilde{c}_0^1 \, T \, c_0^1$ in $\hat{G}$. The distance between $\phi(x)$ and $\phi(c_0^1)$ in G is at most $|V|$, thus there is some $\tilde{c}_0^1 \, T \, c_0^1$ in $\hat{V}$ such that $d(x, \tilde{c}_0^1) \leq |V|$.

The strategy is as follows, see Figure 3. By teleporting to $\tilde{c}_0^1$ and then choosing c_{j+1}^1 as close as possible to x, Cops ensures that $c_j^1 = x$ for some $j \leq |V|$, in particular, she can make sure that $c_{|V|}^1 = x$.

For $j > |V|$ we proceed as follows. Let r'_j be the unique vertex on P at distance $\min(d(u, r_j), D)$ from u. If $c_j^1 = r'_j$, then Cops chooses $c_{j+1}^1 = c_j^1$, otherwise c_{j+1}^1 is the neighbour of c_j^1 on P which lies closer to r'_j than c_j^1. Independence of this strategy from c_j^i for $i \neq 1$ is obvious.

Note that $d(u, r'_j) \leq \min(D, d(u, r_0) + |V|) = d(u, x)$ for $j \leq |V|$, and thus $d(u, r'_{|V|}) \leq d(u, c_{|V|}^1)$. Since r'_{j+1} is contained in the closed neighbourhood of r'_j in P, there must be some $m \geq |V|$ such that for $|V| \leq j < m$ we get that c_{j+1}^1 is the neighbour of c_j^1 which lies closer to u and for $j \geq m$ we get $c_{j+1}^1 = r'_j$.

Clearly, $d(u, r_j) \leq d(u, r_0) + |V|$ for every $j \leq |V|$. Moreover, if $m > |V|$, then $d(u, r_j) = d(u, r'_j) < d(u, c_j^1) \leq d(u, r_0) + |V|$ for $|V| < j < m$. It easily follows that $d(u, r_j) \leq d(u, r_0) + |V|$ for any $j \leq m$. The second property follows from the fact that if $r_j \in P$ for $j \geq m$, then $r_j = r'_j = c_{j+1}^1$. □

The next lemma is already implicit in [1]. We provide a proof for the sake of completeness—essentially this is the same as the proof of [1, Theorem 6], starting in situation (b), described on [1, page 9] which roughly corresponds to condition (∗) below.

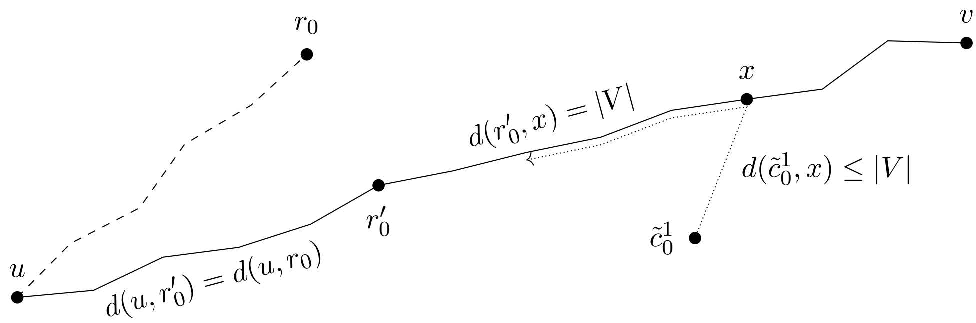


Figure 3: Situation in the proof of Lemma 3.3. The solid line is P, the dotted line indicates, how c^1_j develops after teleportation to $\tilde{c}^1_0$.

The basic idea of the strategy is that Cops always has the robber surrounded by two cops. More formally, Cops will ensure that the following condition is satisfied with respect to some fixed embedding.

($*$) There are paths P and Q (one of which may be empty), a finite component R of $G \setminus (P \cup Q)$, and $m \in \mathbb{N}$ such that

  a) P and Q embed on the boundary of the outer face of the graph induced by $P \cup Q \cup R$,

  b) $r_m \in R$, and

  c) Cops has a strategy such that P is i-guarded and Q is i'-guarded with $i \neq i'$ for every $j \geq m$.

Note that we allow the paths to be empty, that is, we consider the empty graph as a path. The reason for this is that it reduces the casework involved in the proof. It is also worth noting that if G is a finite, connected planar graph, then ($*$) can easily be achieved for a path P consisting of one vertex and an empty path Q. In particular, the lemma below implies that any such graph satisfies $c(G) \leq 3$.

Lemma 3.4. *Let G be a (potentially infinite) connected planar graph with some fixed planar embedding. If ($*$) holds in* Cops and Robber *with* 3 *cops, then* Cops *has a winning strategy.*

Proof. We proceed inductively. After each iteration, unless Cops has won the game, we end up in situation ($*$). The values $|R|$ and $|P|+|Q|+|R|$ never increase in an iteration. Moreover at least one of the two values decreases unless one of the paths is empty before the iteration (in which case both paths are non-empty afterwards). Since $|P|$, $|Q|$, and $|R|$ are non-negative integers this implies that there is some finite upper bound on the number of iterations which means that Cops must eventually win the game.

We now turn to the iterative definition of the strategy. If Q has an endpoint q that either lies on P or is not incident to R, then replace Q by $Q' = Q \setminus \{q\}$. Since Q is

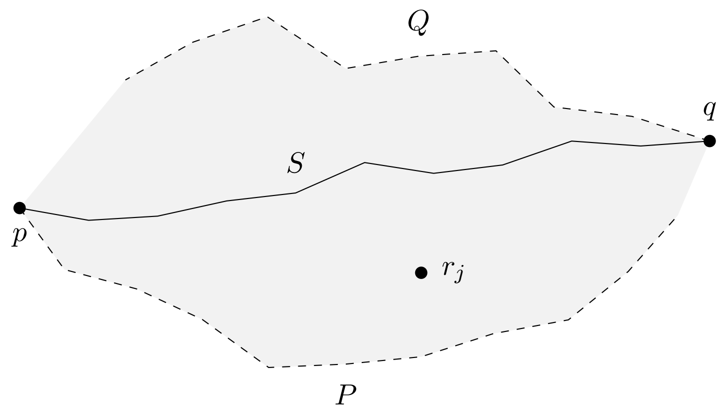


Figure 4: Situation in the proof of Lemma 3.4: R is embedded in the shaded region of $\mathbb{R}^2$, all neighbours of R lie in R, P, or Q. Note that any path in R from r_j to Q must contain a vertex of P or S.

i'-guarded at time $j \geq m$, so is Q'. In particular, P, Q', and R satisfy the conditions of $(*)$. Note that we left R unchanged and decreased the value of $|P| + |Q| + |R|$.

Since P and Q both lie on the boundary of the outer face, they can only intersect if an endpoint of one of them lies on the other. Thus by iterating the above argument (and possibly exchanging the roles of P and Q) we can assume that either Q is empty, or P and Q are both non-empty and disjoint, and all of their endpoints are incident to R.

First assume that Q is the empty path, and without loss of generality assume that P is 1-guarded. If P contains all vertices on the boundary of the outer face of the subgraph induced by $P \cup R$, then let q be an endpoint of P, and let $P' = P \setminus \{q\}$. Otherwise, let q be a vertex that is incident to the outer face but does not lie on P, and let $P' = P$. Let Q' be the path consisting only of the vertex q. Note that in both cases P' and Q' are non-empty (as claimed) because the boundary of the outer face contains at least 3 vertices.

Since P is 1-guarded at time $j \geq m$, so is P'. Moreover, Cops can ensure that for some $m' \geq m$ and all $j \geq m'$ we have that $c_j^2 = q$. If $r_j \in P$ for some $m \leq j \leq m'$ or $r_{m'} = q$, then Cops has won the game. Otherwise, $r_{m'}$ is contained in some component R' of $R \setminus \{q\}$ and P', Q', and R' satisfy $(*)$. Note that $|R'| \leq |R|$ since $R' \subseteq R$ and $|P'| + |Q'| + |R'| = |P| + |Q| + |R|$ since q was contained in exactly one of P and R.

Finally, assume that P and Q are non-empty and disjoint, and their endpoints have neighbours in R. Orient P and Q in the same direction along the boundary of the outer face and let p and q be their first vertices, respectively. Let H' be the subgraph of G induced by p, q, and all vertices in R. If pq is an edge, then let $H = H' \setminus \{pq\}$, otherwise let $H = H'$. Let S be a shortest p-q-path in H—note that such a path exists because R is connected, and both p and q are incident to R. Figure 4 illustrates the resulting situation.

Assume without loss of generality that P is 1-guarded and Q is 2-guarded. If $r_j \in P \cup Q$ for some $j \geq m$, then Cops wins the game. Otherwise $r_j \in R \subseteq H$ for all $j \geq m$, and by

Lemma 3.3 (where T is the equality relation on the vertex set of H) there is a strategy for Cops on H such that for some $m' \geq m$ the paths P, Q, and S are 1-, 2-, and 3-guarded at all times $j \geq m'$, respectively. Clearly, this strategy also works when playing on G: unless $r_j \in P \cup Q$ for some j in which case Robber loses immediately, all of Robber's moves take place in H.

By the above discussion, we already know that $r_{m'} \in R$. If $r_{m'} \in S$, then Cops wins the game since S is 3-guarded. If not, then let R' be the component of $R \setminus S$ containing $r_{m'}$. By Lemma 2.1 and our choice of orientations of P and Q, if the paths $P \setminus \{p\}$ and $Q \setminus \{q\}$ are non-empty, then any path joining $P \setminus \{p\}$ to $Q \setminus \{q\}$ meets S. Since R' is connected and disjoint from S, it thus has neighbours in at most one of $P \setminus \{p\}$ and $Q \setminus \{q\}$. Moreover, R' does not contain any edge connecting the areas bounded by S and the p-q-path on the boundary of H containing P, and by S and the p-q-path on the boundary of H containing Q, respectively; any such edge would either cross an edge of S or contradict the assumption that P and Q lie on the same face. In particular, R' completely embeds into one of these two areas.

Without loss of generality assume that R' has no neighbour in $Q \setminus \{q\}$, and thus R' either embeds into the area bounded by S and the p-q-path on the boundary of H containing P, or R' has no neighbours in $(P \cup Q) \setminus \{p, q\}$. In the former case, $P \setminus \{p\}$, S, and R' satisfy conditions $(*)$. In the latter case, the empty path, S, and R' satisfy them. Since S contains at least one inner point in R, we have $|R'| < |R|$. Furthermore, $P \setminus \{p\}$, S, and R' are disjoint subsets of $P \cup Q \cup R$, whence $|P \setminus \{p\}| + |S| + |R'| \leq |P \cup Q \cup R| \leq |P| + |Q| + |R|$. Clearly, a similar inequality holds with $\emptyset$ instead of $P \setminus \{p\}$, thus completing the proof. □

Theorem 3.5. *If $G = (V, E)$ is a finite toroidal graph, then $c(G) \leq 3$*

Proof. In Definition 3.1, let $\hat{G}$ be a cover of G embedded in the plane as in Lemma 2.2. By a slight abuse of notation we will not distinguish between subgraphs of $\hat{G}$ and their embeddings in $\mathbb{R}^2$. By Lemma 3.2 it suffices to show that $c^T(\hat{G}) \leq 3$.

Assume that Cops and Robber have picked initial positions $(c_0^i)_{i \leq 3}$ and r_0 respectively. Choose D large enough that

$$\frac{D}{|V|} > \log(|B(r_0, D)|),$$

where log denotes the base 2 logarithm and $B(r_0, D)$ is the ball in $\hat{G}$. This is possible because $\hat{G}$ has polynomial growth and V is finite.

Let us call a vertex v a *gateway*, if $d(r_0, v) = D$ and there is an edge connecting v to an infinite component of $\hat{G} \setminus B(r_0, D)$. Define the *finite closure* $\overline{B}$ of $B(r_0, D)$ as the subgraph induced by $B(r_0, D)$ and all finite components of $\hat{G} \setminus B(r_0, D)$. Note that $\hat{G} \setminus \overline{B}$ only has infinite components, and that these are precisely the infinite components of $\hat{G} \setminus B(r_0, D)$. Moreover, any path connecting $\overline{B}$ to $\hat{G} \setminus \overline{B}$ must use a gateway. If a gateway is not incident to the outer face of $\overline{B}$, then the infinite component of $\hat{G} \setminus B(r_0, D)$ it is adjacent to must embed in a bounded face of $\overline{B}$. This bounded face has compact closure, so there would have to be an accumulation point of vertices, contradicting the fact that the embedding from Lemma 2.2 is accumulation free. Hence every gateway lies on the outer face and thus $\overline{B}$ has an embedding in a closed disk such that all gateways lie on the boundary.

Let $(v_i)_{1\leq i\leq l}$ be an enumeration of all gateways in the cyclic order given by their position on the boundary of this disk. For convenience we define $v_0 = v_l$. Note that $l < |B(r_0, D)|$ since all gateways lie at distance D from r_0. Let T be a shortest path tree of $\overline{B}$ rooted at r_0, that is, the unique path in T connecting r_0 to v is a shortest r_0-v-path in $\hat{G}$ for every $v \in \overline{B}$. For $0 \leq i \leq l$, denote by P_i the path from r_0 to v_i in T.

For $a < b$, denote by $[a, b] = \{v_i \mid a < i < b\}$ and $[b, a] = \{v_i \mid i < a \text{ or } i > b\}$. Let $a < b$ and let H be the graph obtained from $\overline{B}$ removing the union of P_a and P_b. By Lemma 2.1 there is no path connecting $[a, b]$ to $[b, a]$ in H.

A vertex $v \in \overline{B}$ lies *between a and b* if it lies in the same component of H as some element of $[a, b]$. By the above observation, no vertex of $\overline{B}$ lies between a and b and between b and a simultaneously. Note that a vertex of $\overline{B}$ lies neither between a and b nor between b and a if and only if each path in $\overline{B}$ connecting it to a gateway contains a vertex of $P_a \cup P_b$, which happens if and only if the vertex is contained in a finite component of $\hat{G} - (P_a \cup P_b)$. Consequently, every vertex of $\overline{B}$ which is contained in an infinite component of $\hat{G} - (P_a \cup P_b)$ must either lie between a and b, or lie between b and a.

We say that Robber is *trapped between a and b* at time j, if r_j lies between a and b, and P_a and P_b are guarded at time j. Note that in this case, Robber will remain trapped between a and b at time $j + 1$ unless either $r_{j+1} \notin B(r_0, D)$, or $r_{j+1} \in P_a \cup P_b$ (in which case Cops wins the game), or Cops changes the strategy and stops guarding P_a or P_b.

We now inductively define for every integer $t \leq \frac{D}{|V|} - 1$ a value $n_t \in \mathbb{N}$, such that one of the following two statements holds.

(I) Cops has won the game before time n_t, or has a strategy to win starting from the position at time n_t.

(II) There are $a_t, b_t \in \mathbb{N}$ such that
   a) $1 \leq b_t - a_t \leq 1 + 2^{-(t+1)} \cdot l$, and
   b) Robber is trapped between a_t and b_t at time $n_t \leq j \leq n_{t+1}$.

Essentially, this is achieved using Lemma 3.3 to inductively guard P_y (for some appropriate y), thus trapping Robber, see Figure 5. We point out that the values n_t are not determined a priori, but depend on how the game evolves. In particular, different strategies of Robber may lead to different values for n_t on the same graph. Throughout the induction, we will also show that $d(r_0, r_{n_t}) \leq (t+1) \cdot |V|$, in order to make sure that $r_{n_t} \in B(r_0, D) \subseteq \overline{B}$.

To start the inductive construction, let $y = \lfloor \frac{l}{2} \rfloor$. By Lemma 3.3 there is a strategy for Cops to make sure that P_0 is 1-guarded at all times $j > m$ and $d(r_0, r_m) \leq |V|$ for some appropriate m. Analogously there is a strategy to make sure that P_y is 2-guarded at all times $j > m'$ and $d(r_0, r_{m'}) \leq |V|$ for some appropriate m'. Since those two strategies

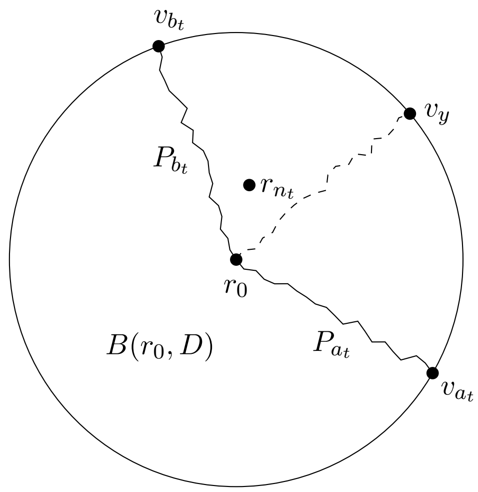


Figure 5: Situation at time j for $n_t \leq j \leq n_{t+1}$: P_{a_t} and P_{b_t} are guarded and Robber is trapped between a_t and b_t. By guarding a shortest r_0–v_y-path P_y (dashed) we make sure that Robber is trapped either between a_t and y, or between y and b_t.

don't interfere with one another, we have a strategy ensuring that both P_0 and P_y are guarded for $j \geq n_0$ and $d(r_0, r_{n_0}) \leq |V|$, where $n_0 := \max(m, m')$.

Let H be the graph obtained from $\hat{G}$ by removing P_0 and P_y. If $r_{n_0} \notin H$, then $r_{n_0} \in P_0 \cup P_y$ and Cops has won the game already. If r_{n_0} is in a finite component of H, then Cops has a winning strategy by Lemma 3.4. In both of these cases (I) holds. Thus we can assume that r_{n_0} lies in an infinite component C of H. Since $r_{n_0} \in B(r_0, |V|) \subseteq \overline{B}$, this vertex must either lie between 0 and y, or it lies between y and l, and thus at time n_0 Robber is trapped either between 0 and y or between y and l. In the first case choose $a_0 = 0$ and $b_0 = y$, in the second case choose $a_0 = y$ and $b_0 = l$. In both cases it is straightforward to check that (II) holds.

For the induction step assume that we have defined n_{t-1} as claimed. If (I) holds, then we can define $n_t = n_{t-1}$ and (I) still holds. So let us assume that (II) holds for n_{t-1}. Let $y = \lfloor \frac{a_{t-1}+b_{t-1}}{2} \rfloor$, let $i \in \{1, 2, 3\}$ be such that neither $P_{a_{t-1}}$ nor $P_{b_{t-1}}$ is i-guarded. Lemma 3.3 provides us with a strategy such that for an appropriate n_t we have that P_y is i-guarded at all times $j \geq n_t$, and $d(r_0, r_{n_t}) \leq d(r_0, r_{n_t}) + |V| \leq (t+1) \cdot |V|$.

Let H be the graph obtained from $\hat{G}$ by removing $P_{a_{t-1}}$, $P_{b_{t-1}}$ and P_y. As before, if $r_{n_t} \notin H$, then Cops has won the game and (I) holds. If r_{n_t} is contained in a finite component of H, then removing two of the three paths from G already leaves it in a finite component (because $\hat{G}$ is planar and $P_{a_{t-1}}$, $P_{b_{t-1}}$ and P_y pairwise don't cross in the embedding). Consequently, Cops has a winning strategy in this situation by Lemma 3.4. Finally assume that r_{n_t} is contained in an infinite component C of H. For $n_{t-1} \leq j \leq n_t$ the paths $P_{a_{t-1}}$ and $P_{b_{t-1}}$ are guarded at time j and $r_j \in \overline{B}$. Together with the assumption that Robber was trapped between a_{t-1} and b_{t-1} at time n_{t-1}, this implies that Robber is trapped between a_{t-1} and b_{t-1} at time n_t unless Cops has won the game before time n_t. Since P_y is also guarded, the same argument as above gives that at time n_t Robber is either trapped between a_{t-1} and y, or between y and b_{t-1}. In the first case take $a_t = a_{t-1}$ and $b_t = y$, in the second case take $a_t = y$ and $b_t = b_{t-1}$. In both cases it is not hard to verify that (II) is satisfied.

To conclude the proof, we remark that the (II) can't possibly be satisfied for $t = \frac{D}{|V|} - 1$. Indeed, in this case

$$2^{-(t+1)} \cdot l < 2^{-\frac{D}{|V|}} \cdot |B(r_0, D)| = 2^{-\frac{D}{|V|} + \log(|B(r_0,D)|)} < 1.$$

Since $b_t - a_t$ is an integer, it follows that $b_t - a_t = 1$, and thus $[a_t, b_t] = \emptyset$. In particular r_{n_t} cannot lie between a_t and b_t. Hence there is some $t \le \frac{D}{|V|} - 1$, such that (I) holds, thus Cops has a winning strategy. □

As mentioned in the introduction, Theorem 3.5 can be used to make progress on Conjecture 1.2. In particular, we have the following result.

Corollary 3.6. *If G is a finite graph of genus $g \le 3$, then $c(G) \le g + 3$.*

We remark that the cases $g \le 2$ were previously known, see [1, 12], so our only real contribution to Corollary 3.6 is the case $g = 3$. We still prove all cases for convenience. We say that a strategy of Cops *reduces the genus by r using s cops*, if it yields i-guarded subgraphs H_i of G for $1 \le i \le s$ such that the genus of the graph obtained from G by removing all H_i is at most $g - s$, where g is the genus of G. Using this notation, we have the following result.

Lemma 3.7. *Assume that we play* Cops and Robber *with $k \ge 4$ cops. Then*

1. Cops *has a strategy reducing the genus by 1 using 2 cops, and*

2. Cops *has a strategy reducing the genus either by 1 using 1 cop, or by 2 using 3 cops.*

Proof. The first part is implicit in [11], the second part is Proposition 3.2 in [12]. □

Proof of Corollary 3.6. For $g = 0$ and $g = 1$ this follows directly from Theorem 3.5 (note that any planar graph can be embedded in the torus). For $g = 2$ apply the first part of Lemma 3.7, then apply Theorem 3.5 to the resulting toroidal graph. For $g = 3$, first apply the second part of Lemma 3.7. If the strategy used 1 cop to reduce the genus by 1, then apply Corollary 3.6 for $g = 2$ to the remaining graph, otherwise apply Theorem 3.5. □

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