On asymmetric colourings of graphs with bounded degrees and infinite motion

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Abstract

A vertex colouring of a graph is called asymmetric if the only automorphism which preserves it is the identity. Tucker conjectured that if every automorphism of a connected, locally finite graph moves infinitely many vertices, then there is an asymmetric colouring with 2 colours. We make progress on this conjecture in the special case of graphs with bounded maximal degree. More precisely, we prove that if every automorphism of a connected graph with maximal degree Δ moves infinitely many vertices, then there is an asymmetric colouring using $\mathcal{O}(\sqrt{\Delta} \log \Delta)$ colours. This is the first improvement over the trivial bound of $\mathcal{O}(\Delta)$.

1 Introduction

Call a (not necessarily proper) vertex colouring of a graph *asymmetric* if the only automorphism which preserves it is the identity. This notion was first introduced by Babai [2] in 1977, who proved that 2 colours suffice for an asymmetric colouring of any regular tree. The concept was later reintroduced by Albertson and Collins [1] under the name *distinguishing* colouring, and since has received a considerable amount of attention.

In this paper we investigate connections between the maximal degree Δ of a graph and the number of colours needed for an asymmetric colouring. It is not hard to see that any connected finite graph has an asymmetric colouring with at most $\Delta + 1$ colours, and it was shown independently by Collins and Trenk [5], and Klavžar, Wong, and Zhou [9]

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that the only graphs which attain this bound are complete graphs, complete bipartite graphs, and the cycle on 5 vertices. In 2007, Imrich, Klavžar and Trofimov [7] extended this result to infinite graphs, showing that any connected infinite graph with maximal degree Δ has an asymmetric colouring with at most Δ colours. In [13], the authors of the present paper improved this bound to $\Delta - 1$ for infinite graphs with maximal degree at least 3 and this bound is sharp for every $\Delta \geq 3$.

It is worth noting that all known examples achieving this bound have an automorphism moving only few vertices. This motivates the concept of *motion*. We say that a graph has motion m, if the minimal number of vertices moved by a non-trivial automorphism is m. Motion turns out to be a powerful tool for bounding the number of colours needed in an asymmetric colouring. It is often used in the form of the so-called Motion Lemma, which states that if G is finite graph with motion m such that $2^{\frac{m}{2}} \geq |\operatorname{Aut} G|$, then G has an asymmetric colouring with 2 colours. Perhaps the first to explicitly state this result were Russell and Sundaram [14], but it is worth pointing out that Cameron, Neumann, and Saxl [3] implicitly use a generalisation to permutation groups to show that primitive permutation groups admit an asymmetric colouring with 2 colours, unless they are symmetric groups, alternating groups, or one of finitely many exceptions. The following conjecture due to Tucker [15] can be seen as a generalisation of the Motion Lemma to infinite graphs.

Conjecture 1.1 (Tucker [15]). *Let G be a connected, locally finite graph with infinite motion. Then there is an asymmetric colouring of G with 2 colours.*

Despite numerous partial results towards this conjecture (see for example [8, 10, 11, 12, 16]) it is still wide open in general. In particular, it is still not known whether the conjecture holds for graphs with finite maximum degree. Recently, Hüning et al. [6] proved that it is true for graphs with maximum degree at most 3, in [13] this was extended to graphs with maximum degree at most 5, and by using similar arguments one can show that there is an asymmetric colouring for connected graphs with infinite motion with at most $\frac{\Delta}{3} + 1$ colours. We note that, although this is a substantial improvement over the trivial bound of $\Delta + 1$ it is still linear in Δ.

In this short note we show that a graph with infinite motion and maximum degree Δ admits an asymmetric colouring with $1 + (\frac{5}{2} + \frac{3}{2} \log_2 \Delta)\lceil \sqrt{\Delta} \rceil = \mathcal{O}(\sqrt{\Delta} \log \Delta)$ colours. To our best knowledge, this is the first such bound which is sublinear in Δ thus constituting a significant step towards Conjecture 1.1 for graphs with finite maximum degree.

2 Definitions and auxiliary results

Throughout this paper G will denote an infinite, connected graph with vertex set V, edge set E, and maximal degree $\Delta \in \mathbb{N}$. Let d denote the usual geodesic distance between vertices of G. For a vertex $v_0 \in V$ denote the *ball* with centre v_0 and radius k by $B(v_0, k) = \{v \in V \mid d(v_0, v) \leq k\}$, and the *sphere* with centre v_0 and radius k by $S(v_0, k) = \{v \in V \mid d(v_0, v) = k\}$. As usual we write $N(v)$ for the set of neighbours of a vertex v.

For a set Ω, denote by $\mathrm{Sym}(\Omega)$ the group of all bijective functions from Ω to itself. Further denote by $\mathrm{Sym}_n := \mathrm{Sym}(\{1, \ldots, n\})$.

Let Γ be a group acting on a set Ω. For $S \subseteq \Omega$, denote the *setwise stabiliser* of S by $\Gamma_S = \{\gamma \in \Gamma \mid \forall s \in S \colon \gamma s \in S\}$ and denote the *pointwise stabiliser* of S by $\Gamma_{(S)} = \{\gamma \in \Gamma \mid \forall s \in S \colon \gamma s = s\}$. Observe that $\Gamma_{(\Omega)}$ is a normal subgroup of Γ, but the same isn't necessarily true for $\Gamma_{(S)}$ for $S \subsetneq \Omega$. In particular, it makes sense to speak about the quotient $\Gamma/\Gamma_{(\Omega)}$, which is the group of all different permutations induced by Γ on Ω. The *stabiliser* of a colouring c of Ω is defined by $\Gamma_c = \{\gamma \in \Gamma \mid \forall v \in \Omega : c(v) = c(\gamma v)\}$. A colouring c of Ω is called *asymmetric* if $\Gamma_c = \Gamma_{(\Omega)}$. Note that in the special case of $\mathrm{Aut}\, G$ acting on G, a vertex colouring is asymmetric if and only if its stabiliser only contains the identity automorphism.

The proof of our main result relies on a corollary to the following result from [4].

Theorem 2.1 (Cameron, Solomon, Turull [4])**.** *The maximal length of a chain of subgroups in Sym_n is $\lfloor \frac{3n-1}{2} \rfloor - b(n)$, where $b(n)$ denotes the number of ones in the binary representation of n.*

Corollary 2.2. *Let Γ be a group acting on two finite sets Ω and Ω' respectively. If $\Gamma_{(\Omega')} \subseteq \Gamma_{(\Omega)}$, then there is $S \subseteq \Omega$ such that $|S| \leq \frac{3|\Omega'|}{2}$, and $\Gamma_{(S)} = \Gamma_{(\Omega)}$.*

Proof. If Γ acts trivially on Ω we can choose $S = \emptyset$. Otherwise pick $s_1 \in \Omega$ with $\Gamma_{(s_1)} \subsetneq \Gamma_{(\Omega)}$. If $\Gamma_{(s_1, s_2, \ldots, s_i)} \neq \Gamma_{(\Omega)}$, we can inductively pick $s_{i+1} \in \Omega$ which is not stabilised by $\Gamma_{(s_1, s_2, \ldots, s_i)}$. This process terminates when $\Gamma_{(s_1, \ldots, s_k)} = \Gamma_{(\Omega)}$.

For $1 \leq i \leq k$, let $\Gamma_i = \Gamma_{(s_1, \ldots, s_i)}$. Recall that $\Gamma_{(\Omega')} \trianglelefteq \Gamma$, and since $\Gamma_{(\Omega')} \subseteq \Gamma_{(\Omega)} \subseteq \Gamma_i$, we also have $\Gamma_{(\Omega')} \trianglelefteq \Gamma_i$. In particular we have a chain

$$\mathrm{Sym}(\Omega') \supseteq \Gamma/\Gamma_{(\Omega')} \supsetneq \Gamma_1/\Gamma_{(\Omega')} \supsetneq \cdots \supsetneq \Gamma_k/\Gamma_{(\Omega')},$$

and the Corollary follows from Theorem 2.1. $\qquad\square$

It is worth noting that Theorem 2.1 (and thus also Corollary 2.2) relies on the classification of finite simple groups. We wish to point out that a slightly weaker bound can be achieved by elementary arguments. Recall that the order of any subgroup divides the order of the ambient group. Since $|S_n| = n! \leq n^n$, any chain of subgroups of S_n has length at most $n \log_2 n$. Using this estimate in the proof of Corollary 2.2 leads to an analogous conclusion with $|S| \leq |\Omega'| \log_2 |\Omega'|$.

3 Proof of the main result

Theorem 3.1. *There is a function $\alpha = \mathcal{O}(\sqrt{\Delta} \log \Delta)$ such that every connected graph with infinite motion and finite maximal degree Δ has an asymmetric colouring with at most α colours.*

Proof. We will inductively define colourings c_k using colours in the set $\{0, 1, 2, 3, \ldots\} \cup \{\bar{1}, \bar{2}, \bar{3}, \ldots\} \cup \{\infty\}$. Note that this set is infinite, however, we will later determine bounds on the number of colours that we actually use.

Choose an arbitrary root $v_0 \in V$. The following properties will be satisfied for every $k \geq 0$.

(a) $c_k(v) = 0$ if and only if $v = v_0$,

(b) $c_k(v) = \infty$ if and only if $d(v, v_0) > k$,

(c) $c_k|_{B(v_0, k-1)} = c_{k-1}|_{B(v_0, k-1)}$ if $k > 0$,

(d) Let Γ_k be the stabiliser of c_k in $\Gamma = \mathrm{Aut}\, G$, then

- $\Gamma_k \leq \Gamma_{(B(v_0, k-1))}$ if $k > 0$,
- all orbits under Γ_k in $B(v_0, k)$ have size at most $\lceil \sqrt{\Delta} \rceil$.

Note that (a) implies that any automorphism which preserves c_k must fix $B(v_0, k)$ setwise, and thus it makes sense to speak of orbits of Γ_k in $B(v_0, k)$.

3.1 Constructing an asymmetric colouring

Before we turn to the construction of the colourings c_k, let us show that they yield an asymmetric colouring of G. By (c) we can define a colouring $c_\infty = \lim_{k \to \infty} c_k$ by $c_\infty|_{B(v_0, k)} = c_k|_{B(v_0, k)}$ for every k. By (a), the only vertex with $c_\infty(v) = 0$ is v_0, hence v_0 is fixed by any automorphism preserving c_∞. Consequently $B(v_0, k)$ is fixed setwise by any colour preserving automorphism. Since $c_\infty|_{B(v_0, k)} = c_{k+1}|_{B(v_0, k)}$, property (d) implies that every automorphism in the stabiliser of c_∞ fixes $B(v_0, k)$ pointwise. This is true for every k whence c_∞ is asymmetric.

3.2 Inductive construction of the colourings

For $k = 0$ let $c_0(v_0) = 0$ and $c_0(v) = \infty$ for $v \neq v_0$. Properties (a) to (d) are trivially satisfied for this colouring.

For the inductive definition of c_{k+1}, assume that we already have defined c_k with the desired properties. By (b) and (c), the only vertices where c_{k+1} and c_k can differ are those in $S(v_0, k+1)$, hence it suffices to describe the colouring c_{k+1} on $S(v_0, k+1)$.

Recall that by (a), the stabiliser Γ_k of c_k must fix $S(v_0, k)$ and $S(v_0, k+1)$ setwise. Let $A_1, \ldots, A_n$ be the orbits of Γ_k on $S(v_0, k+1)$. Define equivalence relations R_i for $0 \leq i \leq n$ on $S(v_0, k+1)$ by

$$x R_i y \iff \forall j \leq i \colon N(x) \cap A_j = N(y) \cap A_j.$$

Note that the condition is void for $i = 0$, hence any two vertices are equivalent under R_0. Further note that R_i has at most one equivalence class of size larger than Δ, namely the class which has no neighbours in A_j for any $j \leq i$. All other equivalence classes are contained in the neighbourhood of some vertex and thus have size at most Δ. This argument also shows that all equivalence classes with respect to R_n have size at most Δ since every vertex in $S(v_0, k+1)$ has a neighbour in $S(v_0, k) = \bigcup_{j \leq n} A_j$.

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Denote by $\mathcal{B}_i$ the set of equivalence classes with respect to the relation R_i. Observe that $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are nested, that is, for every $B \in \mathcal{B}_{i+1}$ there is $B' \in \mathcal{B}_i$ with $B \subseteq B'$.

Let xR_iy and $\phi \in \Gamma_k$. Then $\phi x R_i \phi y$ because ϕ maps neighbours of x and y to neighbours of ϕx and ϕy while fixing every A_j setwise. This shows that Γ_k in its natural action on subsets of $S(v_0, k)$ acts on $\mathcal{B}_i$ by permutations. Denote by $\Gamma_{k,i}$ the pointwise stabiliser of $\mathcal{B}_i$ in Γ_k with respect to this action.

Before we turn to the construction of c_{k+1}, we need one last definition. If $c \colon S(v_0, k+1) \to \mathbb{N}$ is a colouring, then we define the *induced colouring* $c[\mathcal{B}_i] \colon \mathcal{B}_i \to \mathbb{N}$ by

$$c[\mathcal{B}_i](B) = \min_{v \in B} c(v).$$

Note that the stabiliser of any colouring c satisfies $\Gamma_c \subseteq \bigcap_{1 \le i \le n} \Gamma_{c[\mathcal{B}_i]}$.

We now inductively define colourings $c_{k,i}$ of $S(v_0, k+1)$ for $0 \le i \le n$ satisfying the following properties.

(i) $c_{k,i}[\mathcal{B}_j] = c_{k,i-1}[\mathcal{B}_j]$ for $i \ge 1$ and $j < i$.

(ii) Any $B \in \mathcal{B}_i$ is monochromatic under $c_{k,i}$.

(iii) $\tilde{\Gamma}_{k,i} := \bigcap_{j \le i} (\Gamma_k)_{c_{k,i}[\mathcal{B}_j]} \subseteq \Gamma_{k,i}$

Define $c_{k,0} \equiv 1$. The statement (i) is void for $i = 0$, and properties (ii) and (iii) are clearly satisfied.

Now assume that we already defined $c_{k,i}$ with the desired properties. The only way two vertices can be mapped to each other by $\Gamma_{k,i}$ and are not equivalent in R_{i+1} is, if they have different neighbours in A_{i+1}. Hence the stabiliser of A_{i+1} in $\Gamma_{k,i}$ fixes $\mathcal{B}_{i+1}$ pointwise, and in particular $(\Gamma_{k,i})_{(A_{i+1})} \subseteq \Gamma_{k,i+1}$. Since $\tilde{\Gamma}_{k,i} \subseteq \Gamma_{k,i}$ we have $(\tilde{\Gamma}_{k,i})_{(A_{k+1})} \subseteq \tilde{\Gamma}_{k,i} \cap \Gamma_{k,i+1} \subseteq (\tilde{\Gamma}_{k,i})_{(\mathcal{B}_{i+1})}$, and by Corollary 2.2, there is a subset $\mathcal{S} \subseteq \mathcal{B}_{i+1}$ such that

$$|\mathcal{S}| \le \frac{3|A_{k+1}|}{2} \le \frac{3\lceil \sqrt{\Delta} \rceil}{2}$$

and

$$(\tilde{\Gamma}_{k,i})_{(\mathcal{S})} = (\tilde{\Gamma}_{k,i})_{(\mathcal{B}_{i+1})}.$$

Let $\mathcal{S} = \{S_1, \dots, S_\ell\}$ be a minimal subset (with respect to cardinality) with these two properties. For $j \in \{1, \dots, \ell\}$ and each $v \in S_j$ let $c_{k,i+1}(v) = c_{k,i}(v) + j$. For all other vertices set $c_{k,i+1}(v) = c_{k,i}(v)$. Property (ii) for $c_{k,i+1}$ trivially follows from (ii) for $c_{k,i}$. It remains to show that conditions (i) and (iii) hold.

For (i), first note that for any $j \le i$ and $B'' \in \mathcal{B}_j$, there is $B' \subseteq B''$ with $B' \in \mathcal{B}_i$ such that all vertices of B' are coloured with colour $c_{k,i}[\mathcal{B}_j](B'')$. This follows from (ii) for $c_{k,i}$ and the definition of $c_{k,i}[\mathcal{B}_j]$. In order to ensure (i) for $c_{k,i+1}$, we need to ensure that the colours of some vertices in B' do not change. To this end, it is enough to show that there is $B \subseteq B'$ with $B \in \mathcal{B}_{i+1} \setminus \mathcal{S}$.

Assume for a contradiction that every such B is contained in $\mathcal{S}$. Pick $S \in \mathcal{S}$ with $S \subseteq B'$ and let $\mathcal{S}' = \mathcal{S} \setminus \{S\}$. By minimality of $\mathcal{S}$ we know that $(\tilde{\Gamma}_{k,i})_{(\mathcal{S})} \subsetneq (\tilde{\Gamma}_{k,i})_{(\mathcal{S}')}$. In

particular, there must be $\gamma \in (\tilde{\Gamma}_{k,i})_{(\mathcal{S}')}$ which acts non-trivially on S. Since (iii) holds for $\tilde{\Gamma}_{k,i}$ we have that $\gamma S \subseteq B'$, but this contradicts the fact that every other $B \in \mathcal{B}_{i+1}$ with $B \subseteq B'$ is contained in $\mathcal{S}'$ and thus fixed by $(\tilde{\Gamma}_{k,i})_{(\mathcal{S}')}$.

The same argument as above also shows that for $S \in \mathcal{S}$ and $B' \in \mathcal{B}_i$ with $S \subseteq B$ we have that S is not stabilised by $\tilde{\Gamma}_{k,i}$. In particular, S can be mapped to a different subclass of B' and hence $|S| \leq \frac{|B'|}{2}$. This fact will be used later to bound the number of used colours.

For the proof of (iii), note that by definition of $\mathcal{S}$ we have $(\tilde{\Gamma}_{k,i})_{(\mathcal{S})} \subseteq (\Gamma_{k,i})_{(\mathcal{B}_{i+1})} = \Gamma_{k,i+1}$. Further note that by (i) we have $(\Gamma_k)_{c_{k,i}[\mathcal{B}_j]} = (\Gamma_k)_{c_{k,i+1}[\mathcal{B}_j]}$ for every $j \leq i$, and consequently

$$\tilde{\Gamma}_{k,i+1} = \tilde{\Gamma}_{k,i} \cap (\Gamma_k)_{c_{k,i+1}[\mathcal{B}_{i+1}]} = (\tilde{\Gamma}_{k,i})_{c_{k,i+1}[\mathcal{B}_{i+1}]}.$$

So all we need to show is that

$$(\tilde{\Gamma}_{k,i})_{c_{k,i+1}[\mathcal{B}_{i+1}]} \subseteq (\tilde{\Gamma}_{k,i})_{(\mathcal{S})},$$

that is, every element of $\tilde{\Gamma}_{k,i}$ which preserves $c_{k,i+1}[\mathcal{B}_{i+1}]$ must fix $\mathcal{S}$ pointwise.

Let $S \in \mathcal{S}$, let $\gamma \in (\tilde{\Gamma}_{k,i})_{c_{k,i+1}[\mathcal{B}_{i+1}]}$ and let $B \in \mathcal{B}_i$ such that $S \subseteq B$. By (iii) we know that $\gamma \in \Gamma_{i,k}$ and thus $\gamma S \subseteq B$. Since $S \in \mathcal{S}$ we have (by (ii) for $c_{k,i}$ and the definition of $c_{k,i+1}$) that the colour of the vertices of S is different from the vertices in any subclass of B, and hence $\gamma S = S$.

Finally, we obtain c_{k+1} from $c_{k,n}$ as follows. Split every equivalence class in $\mathcal{B}_n$ into at most $\lceil \sqrt{\Delta} \rceil$ subsets of size at most $\lceil \sqrt{\Delta} \rceil$. This is possible because every equivalence class with respect to R_n contains at most Δ vertices. Now for each $B \in \mathcal{B}_n$, one of the subclasses keeps the same colour as in $c_{k,n}$, while the others are recoloured with distinct colours in $\overline{1}, \overline{2}, \ldots, \overline{\lceil \sqrt{\Delta} \rceil}$.

Finally, we need to check that c_{k+1} has the desired properties. By construction, it satisfies (a), (b), and (c). These properties also immediately imply that $\Gamma_{k+1} \subseteq \Gamma_k$, or more precisely that $\Gamma_{k+1} = (\Gamma_k)_{c_{k+1}}$. The colouring c_{k+1} was constructed from $c_{k,n}$ in a way that every $B \in \mathcal{B}_n$ still contains some vertices with the same colour as in $c_{k,n}$. Hence every element of Γ_k which preserves c_{k+1} must also preserve $c_{k,n}$, whence $\Gamma_{k+1} \subseteq \Gamma_{k,n}$ by (iii) for $c_{k,n}$.

In particular, $\Gamma_{k,n}$ setwise fixes all equivalence classes with respect to R_n, and by construction of c_{k+1} from $c_{k,n}$ we conclude that there are no orbits of size more than $\lceil \sqrt{\Delta} \rceil$ in $S(v_0, k+1)$. The second part of (d) follows by induction.

For the first part of (d), assume for a contradiction that there is $\gamma \in \Gamma_{k+1}$ fixing every $B \in \mathcal{B}_n$ but not fixing $B(v_0, k)$ pointwise. Then for any $v \in S(v_0, k+1)$, the neighbours of v and γv in $B(v_0, k)$ coincide. Hence we can define an automorphism γ' of G by

$$\gamma' v = \begin{cases} v & \text{if } v \in B(v_0, k), \\ \gamma v & \text{otherwise.} \end{cases}$$

This contradicts infinite motion, as $\gamma^{-1} \gamma'$ only moves vertices inside $B(v_0, k)$. Thus we have proved that every element of Γ_{k+1} fixes $B(v_0, k)$ pointwise.

3.3 Number of used colours

To determine the number of colours used in the colouring procedure, first note that at most $\lceil \sqrt{\Delta} \rceil$ colours from the set $\{\bar{1}, \bar{2}, \dots\}$ are used. Hence it only remains to determine the number of colours from $\mathbb{N}$ used throughout the procedure.

For this purpose, let v be a vertex with $c_\infty(v) \in \mathbb{N}$. If $v \in S(v_0, k+1)$, then the colour of v is only changed during the construction of $c_{k,i}$, so it suffices to check how large the colour can get during this construction.

By definition $c_{k,0}(v) = 1$. For $0 \le i \le n$, let $B_i \in \mathcal{B}_i$ be the equivalence class of v with respect to R_i. If $c_{k,i}(v) \ne c_{k,i+1}(v)$, we know that $B_{i+1} \in \mathcal{S}$, and that

$$c_{k,i+1}(v) \le c_{k,i}(v) + |\mathcal{S}| \le c_{k,i}(v) + \frac{3\lceil \sqrt{\Delta} \rceil}{2}.$$

Finally we need to determine how many times it can happen that $B_{i+1} \in \mathcal{S}$. Note that the first time this happens, there must be an automorphism moving B_{i+1} to a different element of $\mathcal{B}_{i+1}$, and in particular B_{i+1} cannot be the unique class whose size is larger than Δ. Every subsequent time, we have that B_{i+1} and its image are subclasses of B_i of equal size. Consequently $|B_{i+1}| \le \frac{|B_i|}{2}$, and thus $B_{i+1} \in \mathcal{S}$ for at most $(1 + \log_2 \Delta)$ many i.

Summing up, we get that

$$c_{k+1}(v) = c_{k,n}(v) \le 1 + (1 + \log_2 \Delta) \frac{3\lceil \sqrt{\Delta} \rceil}{2}.$$

Together with the additional $\lceil \sqrt{\Delta} \rceil$ colours from the set $\{\bar{1}, \bar{2}, \dots\}$ we have thus used at most $1 + (\frac{5}{2} + \frac{3}{2} \log_2 \Delta) \lceil \sqrt{\Delta} \rceil = \mathcal{O}(\sqrt{\Delta} \log \Delta)$ colours as claimed. $\qquad\square$

References

[1] M. O. Albertson and K. L. Collins. Symmetry breaking in graphs. *Electron. J. Comb.*, 3(1), 1996.

[2] L. Babai. Asymmetric trees with two prescribed degrees. *Acta Math. Acad. Sci. Hung.*, 29, 1977.

[3] P. Cameron, P. Neumann, and J. Saxl. On groups with no regular orbits on the set of subsets. *Archiv der Mathematik*, 43:295–296, 01 1984.

[4] P. J. Cameron, R. Solomon, and A. Turull. Chains of subgroups in symmetric groups. *J. Algebra*, 127(2):340–352, 1989.

[5] K. L. Collins and A. N. Trenk. The distinguishing chromatic number. *Electron. J. Comb.*, 13(1):research paper r16, 2006.

[6] S. Hüning, W. Imrich, J. Kloas, H. Schreiber, and T. W. Tucker. Distinguishing graphs of maximum valence 3. *Electron. J. Comb.*, 26(4):research paper P4.36, 2019.

[7] W. Imrich, S. Klavžar, and V. Trofimov. Distinguishing infinite graphs. *Electron. J. Comb.*, 14(1):research paper r36, 12, 2007.

[8] W. Imrich, S. M. Smith, T. W. Tucker, and M. E. Watkins. Infinite motion and 2-distinguishability of graphs and groups. *J. Algebr. Comb.*, 41(1):109–122, 2015.

[9] S. Klavžar, T.-L. Wong, and X. Zhu. Distinguishing labellings of group action on vector spaces and graphs. *J. Algebra*, 303(1):626–641, 2007.

[10] F. Lehner. Random colourings and automorphism breaking in locally finite graphs. *Comb. Probab. Comput.*, 22(6):885–909, 2013.

[11] F. Lehner. Distinguishing graphs with intermediate growth. *Combinatorica*, 36(3):333–347, 2016.

[12] F. Lehner. Breaking graph symmetries by edge colourings. *J. Comb. Theory, Ser. B*, 127:205–214, 2017.

[13] F. Lehner, M. Pilśniak, and M. Stawiski. Distinguishing infinite graphs with bounded degrees. *preprint*, arXiv:1810.03932, 2019.

[14] A. Russell and R. Sundaram. A note on the asymptotics and computational complexity of graph distinguishability. *Electron. J. Comb.*, 5:research paper R23, 1998.

[15] T. W. Tucker. Distinguishing maps. *Electron. J. Comb.*, 18(1):research paper p50, 21, 2011.

[16] M. E. Watkins and X. Zhou. Distinguishability of locally finite trees. *Electron. J. Comb.*, 14(1):research paper r29, 10, 2007.