

On spanning tree packings of highly edge connected graphs

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Abstract

We prove a refinement of the tree packing theorem by Tutte/Nash-Williams for finite graphs. This result is used to obtain a similar result for end faithful spanning tree packings in certain infinite graphs and consequently to establish a sufficient Hamiltonicity condition for the line graphs of such graphs.

Keywords: infinite graph theory, end faithful spanning tree, spanning tree packing, Hamiltonian cycle

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1. Introduction

A spanning tree packing of a graph is a set of edge disjoint spanning trees. The following theorem, discovered independently by Tutte and Nash-Williams, provides a sufficient condition for the existence of a spanning tree packing of cardinality k .

Theorem 1.1 (Tutte [28], Nash-Williams [25]). *Every finite $2k$ -edge connected graph has k edge disjoint spanning trees.*

It is known that this result does not remain true for infinite graphs, not even for locally finite graphs, i.e., infinite graphs where every vertex has only finitely many neighbours. Aharoni and Thomassen [1] showed that it is possible to construct locally finite graphs of arbitrarily high edge connectivity that do not possess two edge disjoint spanning trees.

Meanwhile the following approach due to Diestel and Kühn [15, 16] allows a natural extension of the result to locally finite graphs. They proposed to use topological notions of paths and cycles in infinite graphs, which has the advantage of being able to define cycles containing infinitely many edges. More precisely they used homeomorphic images of the unit interval and the unit circle in the Freudenthal compactification of a graph (so called arcs and topological

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circles) as infinite analogues of paths and cycles in finite graphs. They also introduced the concept of a topological spanning tree, which is an infinite analogue of finite spanning trees compatible with the notions of arcs and topological circles.

Using the topological notions above numerous results from finite graph theory have been generalised to locally finite graphs [2, 3, 4, 5, 6, 7, 8, 12, 13, 15, 16, 17, 18, 19, 21, 26], and even to general topological spaces [20, 29], substantiating their impact on infinite graph theory. As mentioned earlier these new concepts can also be used to establish a generalisation of Theorem 1.1. The following result can be found in [14].

Theorem 1.2. *Every locally finite $2k$ -edge connected graph has k edge disjoint topological spanning trees.*

The starting point of this paper was the following conjecture by Georgakopoulos related to the Hamiltonian problem in infinite graphs.

Conjecture 1.3 (Georgakopoulos [17]). *The line graph $L(G)$ of every locally finite 4-edge connected graph G has a Hamiltonian circle.*

By an infinite Hamiltonian circle we mean a topological circle containing every vertex and every end. In the finite case Conjecture 1.3 is known to be true. This was first observed by Thomassen [27] who stated the fact without proof. A simple proof later given by Catlin [9] makes use of Theorem 1.1. However, it turns out that Theorem 1.2 is insufficient to provide a similar proof for Conjecture 1.3. Hence we need a generalisation of the theorem of Tutte/Nash-Williams involving a better notion of spanning trees.

In the present paper we establish a sufficient condition for the existence of large end faithful spanning tree packings similar to Theorem 1.1 where an end faithful spanning tree packing is a spanning tree packing in which every tree is end faithful.

Theorem 1.4. *Let G be a $2k$ -edge connected locally finite graph with at most countably many ends. Then G admits an end faithful spanning tree packing of cardinality $k - 1$.*

We also show that for a graph with at most countably many ends the topological spanning tree packing in Theorem 1.2 can be chosen in a way that the union of any two of the topological spanning trees is an end faithful subgraph of G .

Theorem 1.5. *Let G be a $2k$ -edge connected locally finite graph with at most countably many ends. Then G admits a topological spanning tree packing \mathcal{T} of cardinality k such that $T_1 \cup T_2$ is an end faithful connected spanning subgraph of G whenever $T_1, T_2 \in \mathcal{T}$ and $T_1 \neq T_2$.*

Note that the counterexample of Aharoni and Thomassen [1] mentioned earlier shows that there is a locally finite graph with uncountably many ends which does not permit two edge disjoint connected spanning subgraphs. Hence

Theorems 1.4 and 1.5 are optimal in a sense that they do not hold if we consider graphs with uncountably many ends. Still, it may be possible to improve on the edge connectivity, that is, $2k$ -edge connectivity may be sufficient for the existence of an end faithful spanning tree packing of cardinality k .

In the proof of the two results we use the following non-trivial refinement of Theorem 1.1 for finite graphs which we prove in Section 4.

Theorem 1.6. *Let $G = (V, E)$ be a finite $2k$ -edge connected graph and let $u, v \in V$ be such that $E_u := \{e \in E \mid e \text{ is incident to } u\}$ is a u - v -cut of minimal cardinality. Let a and b be vertices in the same component of $G - \{u, v\}$. Then there is a spanning tree packing \mathcal{T} of cardinality k of $G - u$ and two trees $T, T' \in \mathcal{T}$ such that a and b lie in the same component of $(T \cup T') - v$.*

Using the tree packing results above we establish a sufficient Hamiltonicity condition for line graphs of locally finite graphs. This extends a recent result by Brewster and Funk [2] and partially verifies Conjecture 1.3.

Theorem 1.7. *The line graph of every locally finite 6-edge connected graph with at most countably many ends has a Hamiltonian circle.*

Finally we outline the example of Aharoni and Thomassen [1] and show that it is no counterexample to Conjecture 1.3.

2. Basic Definitions and Facts

All graphs considered in this paper are multigraphs, i.e., we allow multiple edges but no loops. For any notions that are not explicitly defined we will be using the terminology of Diestel [14].

For the remainder of this section let $G = (V, E)$ be a connected (multi)graph.

A set $S \subseteq E$ is called an *edge cut* or simply a *cut*, if $G - S$ is not connected. A cut S *separates* two sets of vertices U and U' if there is no U - U' -path in $G - S$. In this case S is called a U - U' -cut. For convenience we omit the brackets for a $\{u\}$ - $\{u'\}$ -cut.

For sets of vertices $U, U' \subseteq V$ define the *local edge connectivity* $\kappa'_G(U, U')$ as the minimal cardinality of a U - U' -edge cut. We write $\kappa'(U, U')$ instead of $\kappa'_G(U, U')$ if G is clear from the context. The (*global*) *edge connectivity* of G is defined as $\kappa'(G) = \min_{U, U' \subseteq V} \kappa'_G(U, U')$. Note that $\kappa'(G)$ is the minimal number of edges we have to remove to disconnect the graph.

The following well known theorem by Menger provides an alternative characterisation of local and thus also global edge connectivity. We will sometimes use this characterisation without explicitly mentioning it.

Theorem 2.1 (Menger [24]). *Let $G = (V, E)$ be a finite graph and $U, U' \subseteq V$. Then $\kappa'(U, U') \geq k$ if and only if there are k edge disjoint U - U' -paths.*

If $U \subseteq V$ then the *contraction* of U in G is defined by $G/U = (V', E')$ where V' is obtained from V by replacing the set U by a new vertex $x_U \notin V$ and E' is obtained from E by replacing all endpoints of edges in U by x_U and deleting all

loops. We will also call a graph a contraction of G if it can be obtained from G by a sequence of contractions. If $A, B \subseteq V$ such that at most one of the sets has non-empty intersection with U then clearly an A - B -cut of minimal cardinality in G will be taken to an A' - B' -cut of minimal cardinality in G/U where A' and B' are the subsets of V' corresponding to A and B respectively. In particular contractions do not decrease edge connectivity.

The *subgraph induced* by U in G is defined as $G[U] = (U, E \cap U^2)$.

The *minor induced* by U in G is the graph obtained from G by contracting every component of $G - U$ to a vertex and denoted by $G\{U\}$.

Let $H = (V', E')$ be a subgraph of G and let G' be a contraction of G . The *restriction* of H to G' is defined as the subgraph of G' that contains exactly the edges corresponding to edges of H and is denoted by $H|_{G'}$.

Let s be a vertex of degree ≥ 2 . If u and v are neighbours of s , *splitting off* the pair of edges $\{us, vs\}$ means deleting these two edges and replacing them by a new edge uv .

The inverse operation of splitting off is called *pinching*, i.e., pinching a set of edges $E' \subseteq E$ at a vertex w means replacing every edge $uv \in E'$ by the two edges uw and vw . Note that if $w \notin V$ we need to add w to V first. For convenience we will omit the brackets if $E' = \{e\}$.

The following theorem by Mader will be an important ingredient to the proofs of our main results. A bridge here means a cut of cardinality one, i.e., an edge whose removal disconnects the graph (note that such an edge cannot exist if $\kappa'(G) > 1$).

Theorem 2.2 (Mader [23]). *Let $G = (V, E)$ be a finite connected graph, $s \in V$ not incident to any bridges in G and $\deg(s) \neq 3$. Then we can find a pair of edges incident to s that can be split off such that the local edge connectivity remains unchanged for all $x, y \in V \setminus \{s\}$.*

A *ray* is a one-sided infinite path, the subrays of a ray is called its *tails*. We call two rays *equivalent* if there is no finite cut S such that tails of the two rays lie in different components of $G - S$. The equivalence classes with respect to this relation are called the *ends* of G , the set of ends is denoted by $\Omega(G)$. An ω -*ray* is a ray contained in the equivalence class $\omega \in \Omega$. While technically rays are paths, the word path will mean a finite path unless explicitly stated otherwise.

Endow G with the topology of a 1-complex, i.e., every edge is homeomorphic to the real unit interval with any two of them being internally disjoint. Edges that share a vertex are glued together at one endpoint. We can extend this topology to $G \cup \Omega(G)$ by defining basic open neighbourhoods of $\omega \in \Omega$ in the following way: let S be a finite cut and denote by C_S the component of $G - S$ in which all ω -rays lie. Let Ω_S be the set of ends containing a ray in C_S and denote by E_S a set containing a half open edge of every edge connecting C_S to $G - C_S$. Then $C_S \cup \Omega_S \cup E_S$ is a basic open neighbourhood of ω . The arising topological space is called the *Freudenthal compactification* or *end compactification* of G and denoted by \overline{G} . The set Ω of ends of G with the induced topology is called the *end space* of G .

There are some ambiguous notions appearing in both topology and graph theory, like connectedness, which may not coincide for a subgraph H of G and its closure in \overline{G} . Throughout this paper these notions will refer to graph theoretical concepts. If we use the topological notions we will state it explicitly.

An *infinite star* is the complete bipartite graph $K_{1,|\mathbb{N}|}$. A *subdivision* of a graph is obtained from this graph by replacing every edge by a finite path. An *infinite comb* is a graph consisting of a ray γ and infinitely many disjoint paths having exactly their first vertex on γ . The other endpoints of the paths are called the *teeth* of the comb, γ is called its *spine*. Note that a tooth may lie on the spine if the respective path has length zero. The following lemma is a standard result in infinite graph theory, for a proof see Diestel [14].

Lemma 2.3 (Star-Comb-Lemma). *Let $U \subseteq V$ be an infinite set of vertices of G . Then G contains either a subdivision of an infinite star with all leaves in U or an infinite comb with all teeth in U .*

Clearly in a locally finite graph always the latter is the case. Another easily observed fact is that the sequence of the teeth of a comb converges to the end in which its spine lies.

Let H be a subgraph of G . Then there is a unique continuous function $\varphi: \overline{H} \rightarrow \overline{G}$ such that $\varphi|_H = \text{id}$. We say that H is *end faithful* if $\varphi|_{\Omega(H)}$ is a homeomorphism of the end spaces of H and G . Equivalently, H is an end faithful subgraph if it contains an ω -ray for every end ω of G and any two rays in H which lie in the same end of G also lie in the same end of H . Note that a spanning tree T of G is end faithful if and only if for every end ω of G any two ω -rays in T have a common tail.

A *topological path* is a continuous (but not necessarily injective) map from the closed unit interval $[0, 1]$ to \overline{G} . The image of an injective topological path is called an *arc*.

We call a homeomorphic image of the half open unit interval $[0, 1)$ in \overline{G} a *topological ray*. Analogous to ordinary rays we call the image of $[a, 1)$ for $a \in [0, 1)$ a *topological tail* of that ray. We say that a topological ray γ *converges to* $x \in \overline{G}$ if defining $\gamma(1) = x$ would make it a topological path. Notice that all of its topological tails converge to x as well. Usually x will be an end of G .

A homeomorphic image of the unit circle C^1 in \overline{G} is called a *topological circle*.

A topologically path-connected subspace of \overline{G} that does not contain a topological circle is called a *topological tree*. A *topological spanning tree* of G is a topological tree that contains all vertices and all ends of G and every edge of which it contains an inner point.

A (*topological*) *spanning tree packing* of G is a set of pairwise edge disjoint (topological) spanning trees of G .

For the sake of simplicity we will not distinguish between a subgraph of G and its closure in \overline{G} , for example, we will call a subgraph T of G a topological spanning tree of G if its closure is a topological spanning tree in \overline{G} .

3. Spanning Trees and Topological Spanning Trees

The following section contains a collection of facts about spanning trees and topological spanning trees that we will use throughout this paper.

Proposition 3.1. *Let $G = (V, E)$ be a locally finite graph, $U \subseteq V$ and let $\{U_i \mid i \in I\}$ be a partitioning of $V \setminus U$ such that $G[U_i]$ is connected and there are only finitely many edges connecting U_i to $V \setminus U_i$ for every i . Denote by G_U the graph obtained from G by contracting every U_i . Given spanning trees T_U of G_U and T_i of $G[U_i]$ there is a spanning tree T of G satisfying $T|_{G_U} = T_U$ and $T[U_i] = T_i$.*

The statement still holds if we replace spanning trees by topological spanning trees.

The proof is easy and straightforward and will be left to the reader. Note that since G is locally finite I is at most countable. If we start with sets of k edge disjoint spanning trees of each of the $G[U_i]$ and of G_U then by Proposition 3.1 we can find k edge disjoint spanning trees of G .

Definition 3.2. Let $G = (V, E)$ be a graph. A non-decreasing sequence $(V_n)_{n \in \mathbb{N}}$ of subsets of V is called *exhausting* if $\lim_{n \rightarrow \infty} V_n = V$.

Lemma 3.3. *Let $G = (V, E)$ be a locally finite graph and let V_n be an exhausting sequence of subsets of V . For every $n \in \mathbb{N}$ let $\{U_i^n \mid i \in I\}$ be a partitioning of $V \setminus V_n$ such that for every i the graph $G[U_i^n]$ is connected and there are only finitely many edges connecting U_i^n to $V \setminus U_i^n$. Let G_n be the graph obtained from G by contracting every U_i^n and assume that for every U_i^n there is a U_j^{n-1} such that $U_i^n \subseteq U_j^{n-1}$, i.e., G_{n-1} can also be seen as a contraction of G_n . If T_n is a topological spanning tree of G_n and $T_{n+1}|_{G_n} = T_n$ for every $n \in \mathbb{N}$ then $T := \lim_{n \rightarrow \infty} T_n \cup \Omega(G)$ is a topological spanning tree of G .*

Proof. The limit exists because the edge sets of T_n form a non-decreasing sequence of subsets of E and V_n is exhausting.

We claim that there is a topological u - v -path $P \subseteq T$ for every pair $u, v \in V$. Georgakopoulos [19] showed that, given a sequence τ_n of topological u - v -paths there is a subsequence τ_{n_i} such that $\liminf \tau_{n_i}$ contains a topological u - v -path. If we choose each τ_n in a way that its intersection with $G[V_n]$ is contained in T_n then $\liminf \tau_{n_i}$ is contained in T .

If one or both of u and v are ends a similar argument works. Just observe that Georgakopoulos' result also holds for ends. If G_n contains a topological ray to an end then so does T_n and thus we can choose the topological paths τ_n as required.

Finally assume that T contains a topological circle C . Since a circle cannot consist entirely of ends C has to contain at least one vertex, say v . Then v is included in every V_n from some index n_0 on. The restriction of C to T_n contains a topological circle for every $n \geq n_0$, a contradiction to T_n being a tree. \square

Note that unlike Proposition 3.1 the above result does not remain true if we substitute spanning trees for topological spanning trees. The reason for this is that the limit of a sequence finite paths need not necessarily be a finite path. In fact it may happen that such a limit is not even connected.

The last result in this section has already been mentioned in the introduction. It provides a characterisation of the end faithful spanning trees of a graph.

Theorem 3.4 (Diestel and Kühn [16]). *If G is locally finite, then a spanning tree of G is end faithful if and only if its closure in \overline{G} is a topological spanning tree of G .*

4. Spanning Tree Packings in Finite Graphs

In this section we prove Theorem 1.6 and derive several corollaries of this result. It turns out to be easier to prove a strengthening of this theorem, where a and b are allowed to be inner points of edges as well. In order to formulate this stronger version of Theorem 1.6 we need to introduce some notation.

Definition 4.1. Let $G = (V, E)$ be a graph, $v \in V$ and let a and b be vertices or inner points of edges of G . An a - b -arc is called an a - b -bypass of v if it does not contain v .

The set of all spanning tree packings \mathcal{T} of G of cardinality k such that there are l trees in \mathcal{T} whose union contains an a - b -bypass of v will be denoted by $\mathfrak{T}_G^{k,l}(a, b, v)$.

Using the above definition we can state the following lemma which clearly implies Theorem 1.6.

Lemma 4.2. *Let $G = (V, E)$ be a finite $2k$ -edge connected graph and let $u, v \in V$ be such that $E_u := \{e \in E \mid e \text{ is incident to } u\}$ is a u - v -cut of minimal cardinality. Let a and b be vertices or inner points of edges of $G - u$. If there is an a - b -bypass of v in $G - u$ then*

$$\mathfrak{T}_{G-u}^{k,2}(a, b, v) \neq \emptyset. \quad (*)$$

We will prove the statement by induction on the number of vertices. Before doing so however, let us take a look at some consequences of this result. First of all let us restate Theorem 1.6 which is easily seen to be a special case of Lemma 4.2 where a and b are only allowed to be vertices.

Theorem 1.6. *Let $G = (V, E)$ be a finite $2k$ -edge connected graph and let $u, v \in V$ be such that $E_u := \{e \in E \mid e \text{ is incident to } u\}$ is a u - v -cut of minimal cardinality. Let a and b be vertices in the same component of $G - \{u, v\}$. Then there is a spanning tree packing \mathcal{T} of cardinality k of $G - u$ and two trees $T, T' \in \mathcal{T}$ such that a and b lie in the same component of $(T \cup T') - v$.*

Remark. Note that in particular Theorem 1.6 implies that $G - u$ has a spanning tree packing of cardinality k —simply let $a = b \neq v$ if G has at least three vertices. If $G - u$ consists only of v then the existence of such a spanning tree packing is trivial.

The next corollary that we state will be used in the proof of Theorem 1.4.

Corollary 4.3. *Let $G = (V, E)$ be a finite $2k$ -edge connected graph, $k \geq 2$, and let $u, v \in V$ be such that E_u is a u - v -cut of minimal cardinality. Let a and b be vertices of $G - u$. If there is an a - b -arc that does not contain u and v then $\mathfrak{T}_{G-u}^{k-1,1}(a, b, v) \neq \emptyset$.*

Proof. By Theorem 1.6 we can find $\mathcal{T} = \{T^1, T^2, \dots, T^k\} \in \mathfrak{T}_{G-u}^{k,2}(a, b, v)$. We may without loss of generality assume that $T^1 \cup T^2$ contains an a - b -bypass of v . Since both a and b are vertices every a - b -arc is a simple graph theoretical path and thus cycle free.

Every acyclic subgraph of $T^1 \cup T^2$ can be extended to a spanning tree T' of $T^1 \cup T^2$ which is also a spanning tree of $G - u$. Clearly T' contains the same a - b -bypass of v as $T^1 \cup T^2$ and T' and T^i are edge disjoint for $i > 2$ because $T' \subseteq T^1 \cup T^2$. So $\mathcal{T}' := \{T', T^3, \dots, T^k\} \in \mathfrak{T}_{G-u}^{k-1,1}(a, b, v)$. \square

Remark. An analogous proof can be given if a and b are inner points of edges e_a and e_b as long as at least one of the edges is not incident to v . If both e_a and e_b are incident to v then an a - b -bypass of v induces a circle and hence such a bypass cannot be contained in a tree.

The other corollaries in this section will not be used later in the paper, but are merely included as further examples of uses of Theorem 1.6

Corollary 4.4. *Let $G = (V, E)$ be a finite $2k$ -edge connected graph, let $u, v \in V$ and let S be a u - v -cut of minimal cardinality. Then there is a spanning tree packing \mathcal{T} of cardinality k of G such that every tree $T \in \mathcal{T}$ contains exactly one edge in S .*

Proof. Let C_u and C_v be the vertex sets of the two components of $G - S$ in which u and v lie respectively (since S is a minimal cut these are the only components).

Consider the graph G/C_u and denote by u' the vertex obtained from C_u in G/C_u . Clearly $S = \{e \in E(G/C_u) \mid u' \in e\}$ is a u' - v -cut of minimal cardinality and we can apply Theorem 1.6 in order to obtain k edge disjoint spanning trees of $(G/C_u) - u' = G[C_v]$. An analogous argument yields k edge disjoint spanning trees of $G[C_u]$.

By connecting a spanning tree of $G[C_v]$ and a spanning tree of $G[C_u]$ with an edge in S we obtain a spanning tree of G that uses exactly one edge of S . There are at least $2k$ edges in S and we have k edge disjoint spanning trees of $G[C_u]$ and $G[C_v]$ respectively. This allows to construct the desired spanning tree packing and completes the proof. \square

The last corollary we would like to mention, which is sometimes attributed to Catlin [10, 11], characterises the edge connectivity of a graph by the spanning tree packing number of certain subgraphs. It clearly constitutes a refinement of Theorem 1.1.

Corollary 4.5. *Let $G = (V, E)$ be a finite graph.*

1. G is $2k$ -edge connected if and only if $G - F$ admits a spanning tree packing of cardinality k for every set F of at most k edges.
2. G is $(2k + 1)$ -edge connected if and only if $G - F$ admits a spanning tree packing of cardinality k for every set F of at most $k + 1$ edges.

Proof. 1. Necessity is easily seen. For if we remove k edges we still get k edge disjoint spanning trees. Thus we have to remove at least another k edges in order to disconnect the graph. So the minimal cardinality of a cut has to be at least $2k$.

To show that the condition is sufficient let $F \subseteq E$ be a set of k edges and denote by G' the graph obtained from G by pinching F at a vertex $v \notin V$.

G' is $2k$ -edge connected because every cut contains at least $2k$ edges:

- For any two vertices $a, b \in V(G)$ there are $2k$ edge disjoint a - b -paths inherited from G . So the cardinality of any cut that separates a and b is at least $2k$.
- The only cut which does not disconnect two vertices in $V(G)$ is the cut E_v which contains $2k$ edges.

It follows immediately that E_v has to be a v - x -cut of minimal cardinality for an arbitrary vertex $x \in V$ because it is a cut of cardinality $2k$ in a $2k$ -edge connected graph. Hence by Theorem 1.6 we can find k edge disjoint trees of $G' - v = G - F$.

2. The proof of necessity is analogous to the first part.

Now let E' be a set of $k + 1$ edges and $e' \in E'$. Then $G - e'$ is $2k$ -edge connected and thus $(G - e') - (E' \setminus e') = G - E'$ has k edge disjoint spanning trees which proves sufficiency. \square

In the remainder of this section we will prove Lemma 4.2.

Proof of Lemma 4.2. As mentioned earlier we will prove this lemma by induction on the number of vertices. If G is a graph on two vertices then $G - u$ consists only of v . So there cannot be two points a and b as claimed in the condition of Lemma 4.2. Hence induction starts at $|V| = 3$.

Let G be a $2k$ -edge connected graph on three vertices u, v and w and let E_u be a u - v -cut of minimal cardinality.

A spanning tree of $G - u$ consists of a vw -edge. So in order to obtain a spanning tree packing of cardinality k we need to ensure that there are at least k such edges. By Menger's Theorem 2.1 there are as many edge disjoint u - v -paths as edges in E_u . In particular there is a vw -edge for every uw -edge and since $\deg(w) \geq 2k$ there are at least k edges connecting v and w .

Now a and b can be either inner points of vw -edges or equal to w . Either way, there are two vw -edges whose union contains an a - b -bypass of v . We can select these two edges to form trees of the spanning tree packing.

For the induction step we may assume that $|E_u| > \kappa'(G)$ because Lemma 4.2 holds for G if and only if it holds for the graph obtained from G by adding a bundle of parallel uv -edges. Just observe that those edges are irrelevant for the statement of the lemma.

Now let S be a cut such that $|S| = \kappa'(G)$. The cut S does not separate u and v since E_u is a u - v -cut of minimal cardinality and $|E_u| > \kappa'(G)$. Denote by C and C' the vertex sets of the components of $G - S$ and assume without loss of generality that $u, v \in C$.

We will distinguish the following two cases in both of which we will show that (*) holds:

Case 1: $|C'| > 1$.

Case 2: $|C'| = 1$, i.e., $C' = \{w\}$ for some $w \in V$.

In case 1 consider the graph G/C' . Denote by $x_{C'}$ the vertex in G/C' that has been obtained by contracting the set C' and define $a' = x_{C'}$ if a lies in $G[C']$ and $a' = a$ otherwise. Analogously define b' from b .

We now claim that

- (1) $G[C']$ has k edge disjoint spanning trees,
- (2) $\mathfrak{T}_{(G/C')-u}^{k,2}(a', b', v) \neq \emptyset$, and
- (3) the statement (*) follows from (1) and (2).

So let us first prove (1). For this purpose denote by x_C the vertex that corresponds to C in G/C . It can easily be seen that $G[C'] = (G/C) - x_C$. Moreover G/C is $2k$ -edge connected and the set of edges incident to x_C in G/C is a cut of minimal cardinality in G/C . Since $|V| \geq 4$ and $u, v \in C$, G/C has strictly less vertices than G and we can by our induction hypothesis find k edge disjoint spanning trees of $(G/C) - x_C$. This proves (1).

Next we prove (2). Clearly G/C' has strictly less vertices than G , G/C' is $2k$ -edge connected and the set of edges incident to u in G/C' is a u - v -cut of minimal cardinality in this graph. The a - b -bypass of v in $G - u$ corresponds to an a' - b' -bypass of v in $(G/C') - u$. So $\mathfrak{T}_{(G/C')-u}^{k,2}(a', b', v) \neq \emptyset$ by the induction hypothesis. This completes the proof of (2).

In order to prove (3) let $\mathcal{T}_{C'} = \{T_{C'}^1, T_{C'}^2, \dots, T_{C'}^k\}$ be a spanning tree packing of $G[C']$ and let $\mathcal{T}_{G/C'} = \{T_{G/C'}^1, T_{G/C'}^2, \dots, T_{G/C'}^k\} \in \mathfrak{T}_{(G/C')-u}^{k,2}(a', b', v)$. Since we can permute the trees in the packing freely we may without loss of generality assume that $T_{G/C'}^1 \cup T_{G/C'}^2$ contains an a' - b' -bypass of v .

If a lies in $G[C']$ we may assume that a lies in $T_{C'}^1$, for the following reasons. If a is a vertex it is clearly contained in every $T_{C'}^i$. If it is an inner point of an edge contained in some $T_{C'}^i$, we can swap $T_{C'}^1$ and $T_{C'}^i$. Finally, if none of the previous cases holds then a is an inner point of an edge e not contained in any of the $T_{C'}^i$. In this case we can modify $T_{C'}^1$ by adding e and removing an arbitrary edge of the circle that has been closed by doing so.

By an analogous argument we may assume that if b lies in $G[C']$ and if b is not an inner point of an edge of $T_{C'}^1$, then b lies in $T_{C'}^2$.

Now let T^i be the subgraph of G that is obtained by replacing $x_{C'}$ in $T_{G/C'}^i$ by $T_{C'}^i$. We claim that $\mathcal{T} = \{T^i \mid 1 \leq i \leq k\} \in \mathfrak{T}_G^{k;2}(a, b, v)$. By Proposition 3.1 \mathcal{T} is a spanning tree packing. Thus it suffices to prove that $T^1 \cup T^2$ contains an a - b -bypass of v .

- If both a and b lie in $G[C']$ let $x \in C'$ be an arbitrary vertex. Since x is a vertex it is contained in every $T_{C'}^i$. By our choice of $\mathcal{T}_{C'}$ it holds that $a, b \in T_{C'}^1 \cup T_{C'}^2$. Hence we can find an a - x -arc and an x - b -arc in $T_{C'}^1 \cup T_{C'}^2$.

The union of these two arcs clearly contains an a - b -arc. This arc does not contain v because $v \notin C'$, so it is an a - b -bypass of v in $T^1 \cup T^2$.

- If only one of a and b , say a , is contained in $G[C']$ then $T_{G/C'}^1 \cup T_{G/C'}^2$ contains an $x_{C'}$ - b -bypass of v . This implies that there is a vertex $x \in C'$ such that $T^1 \cup T^2$ contains an x - b -bypass B of v .

For the same reason as before there is an a - x -arc A in $T_{C'}^1 \cup T_{C'}^2$.

The union $A \cup B$ contains an a - b -arc which is an a - b -bypass of v because v is not contained in either of A and B .

- If both a and b are not contained in $G[C']$ there is an a - b -bypass of v in $T_{G/C'}^1 \cup T_{G/C'}^2$.

If this bypass does not contain $x_{C'}$ then it is also an a - b -bypass of v in $T^1 \cup T^2$ and we are done.

So assume that it does contain $x_{C'}$. In this case there are vertices $x_a, x_b \in G[V']$ such that $T^1 \cup T^2$ contains an a - x_a -bypass A and an x_b - b -bypass B of v . Furthermore there is an x_a - x_b -arc A' in $T_{C'}^1$, since $T_{C'}^1$ is connected and both x_a and x_b are vertices and thus contained in $T_{C'}^1$.

Clearly $A \cup B \cup A'$ contains an a - b -bypass of v .

This completes the proof of (3) and thus (*) holds in case 1.

Now consider case 2, i.e., assume that $C' = \{w\}$. If $\deg(w) = \kappa'(G)$ is odd then G is $(2k + 1)$ -edge connected. Hence removing an arbitrary edge incident to w from G will leave the graph $2k$ -edge connected. We would like to apply the induction hypothesis later, so we want to select the edge that we delete in a way that E_u remains a u - v -cut of minimal cardinality after deleting it.

To decide which edge to delete choose $\kappa'_G(u, v)$ edge disjoint u - v -paths. The total number of edges incident to w used by these paths has to be even as a path that enters w via one edge has to leave the vertex via another one. Thus there is at least one such edge, say e , which is not used by any of the paths. Hence E_u will still be a u - v -cut of minimal cardinality after e has been removed since there are still $\kappa'_G(u, v)$ edge disjoint u - v -paths.

Now assume that $\deg w$ is even (possibly after deleting an edge incident to w). By Mader's Theorem 2.2 we can split off all edges incident to w in pairs

without changing the local edge connectivity of any pair of vertices in $V - w$. In particular the set of edges incident to u remains a u - v -cut of minimal cardinality throughout this procedure. We then delete the—now isolated—vertex w and denote the graph that we obtain by $H = (V_H, E_H)$. Now define a multiset V' over V_H (i.e., V' consists of vertices but may contain the same vertex more than once) and $E' \subseteq E_H$ as follows:

- Add a copy of $v' \in V_H$ to V' for every $v'w$ -edge that has been split off with a uw -edge. Add another copy of v' if a $v'w$ -edge has been deleted in order to make $\deg(w)$ even.
- Add $e' \in E_H$ to E' if it has been created by splitting off a pair of edges none of which is incident to u .

Note that $|V'| + |E'| \geq k$ because every vertex in V' and every edge in E' corresponds to at most two edges incident to w and $\deg w \geq 2k$. We will need this fact later. It is also an easy observation that $G - u$ is obtained from $H - u$ by pinching E' at w and adding a $v'w$ -edge for every $v' \in V'$. We now claim that

(4) whenever $\mathcal{T}_H = \{T_H^1, T_H^2, \dots, T_H^k\}$ is a spanning tree packing of $H - u$ and e_1, e_2 are edges of $G - u$ not incident to w that did not result from pinching an edge in $\bigcup_{i=3}^k T_H^i$ then there is a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G - u$ such that

- (a) for every $e \in E \cap E_H$ it holds that $e \in T_H^i$ if and only if $e \in T^i$, and
- (b) e_1 and e_2 are contained in $T^1 \cup T^2$.

The first step in the proof of (4) is to turn \mathcal{T}_H into a tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G - u$ by the following pinching procedure, where a tree packing is a set of edge disjoint but not necessarily spanning trees in G .

Begin with $T^i = T_H^i$ for all i then pinch one edge in E' after the other at w and modify the T^i as follows. Let $e = xy$ be the next edge to be pinched.

- (A) If e belongs to none of the T_H^i we do not modify any T^i .
- (B) If $e \in T_H^i$ and no edges incident to w have been added to T^i so far then we remove e from T^i and add both xw and yw to T^i in order to replace e .
- (C) If $e \in T_H^i$ and we have already added edges to T^i before we also remove e from T^i . In this case adding both xw and yw to T^i would result in a circle containing w because there is either a x - w -path or a y - w -path in T^i that does not use e . Hence we only add the edge which is not contained in this circle to T^i .

Note that the T^i remain trees in each of these steps and that no $e \in E \setminus E'$ is removed from or added to T^i . So after pinching all of E' according to (A), (B) and (C) we obtain k edge disjoint trees each of which spans V_H and fulfils property (a).

Still some of the trees may not contain w and thus not be spanning trees. Before dealing with this problem, however, we will take care of property (b).

The edge e_1 can only be contained in T^i for some i if it has been added to T^i applying (B) or (C). Since e_1 did not result from pinching an edge of T^i for $i > 2$ this implies that either $e_1 \in T^1 \cup T^2$ (without loss of generality $e_1 \in T^1$) or e_1 is not contained in any of the T^i at all. In the first case we do not modify any of the T^i . In the latter case, if $w \notin T^1$ adding e_1 to T^1 makes T^1 a spanning tree of $G - u$. If $w \in T^1$ then T^1 is a spanning tree of $G - u$ already. Adding e_1 to it completes a circle that contains w . Remove the other edge incident to w in this circle from T^1 to obtain a tree again. If $e_2 \notin T^1 \cup T^2$ we can use the same procedure as above to obtain $e_2 \in T^2$. Since in this case only T^2 is modified e_1 remains in T^1 .

In the above modifications we did not add or remove any edges in $E \cap E_H$ to any tree. So property (a) still holds.

Finally we need to ensure that all of the trees contain w . For this purpose define

$$\begin{aligned} L &= \{T^i \in \mathcal{T} \mid w \notin T^i\}, \\ L' &= \{e \in E_w \mid u \notin e \text{ and } \forall T^i \in \mathcal{T} : e \notin T^i\}, \end{aligned}$$

and let $l = |L|$ and $l' = |L'|$. All that is left to prove is that $l' - l \geq 0$ because in this case we can add w to each tree in L using an edge in L' .

Note that $l' - l$ is invariant under the above procedure for adding e_1 and e_2 to $T^1 \cup T^2$ because if $w \notin T^1$ both values decrease by one while in the case that $w \in T^1$ they are constant.

So it is sufficient to show that $l' - l \geq 0$ held before these modifications. At that time there is an edge incident to w in $G - u$ which is not used by any tree for every vertex in V' and for every edge in E' for which (B) is not applied. Since (B) is applied exactly once per tree containing w this implies that

$$\begin{aligned} l' &= |L'| \\ &= |V'| + (|E'| - |\mathcal{T} \setminus L|) \\ &= (|V'| + |E'|) - (|\mathcal{T}| - |L|) \\ &\geq k - (k - l) \\ &= l. \end{aligned}$$

This completes the proof of (4).

We will now distinguish the following subcases of case 2 in each of which we will apply the induction hypothesis and (4) to show that (*) holds. Note that we can apply the induction hypothesis to H because H has strictly less vertices than G and the set E_u of edges incident to u is a u - v -cut of minimal cardinality in H .

Case 2a: both a and b lie in H and there is an a - b -bypass of v in $H - u$.

Case 2b: both a and b lie in H and there is no a - b -bypass of v in $H - u$.

Case 2c: a lies in H and b is an inner point of an edge that has been split off to generate an edge $e' \in E'$.

Case 2d: a lies in H and b is an inner point of a $v'w$ -edge for $v' \in V'$.

Case 2e: both a and b are inner points of edges incident to w or are equal to w .

Clearly these cases exhaust all possibilities where a and b could lie. So all we need to show is that $(*)$ holds in each of them.

In case 2a we can apply the induction hypothesis to find a spanning tree packing $\mathcal{T}_H = \{T_H^1, T_H^2, \dots, T_H^k\} \in \mathfrak{T}_{H-u}^{k,2}(a, b, v)$. We may without loss of generality assume that there is an a - b -bypass A of v in $T_H^1 \cup T_H^2$. If A contains no edge of E' then it is also an a - b -bypass of v in any spanning tree packing \mathcal{T} obtained by (4).

So assume that there is at least one edge in E' that is used by A . Denote by e_a the first edge in E' that we pass through when we traverse A starting at a . Clearly e_a is contained in $T_H^1 \cup T_H^2$. It is also easy to see that for one endpoint a' of e_a there is an a - a' -subarc A_a of A that does not contain any inner point of an edge in E' . Analogously define e_b and find a b - b' -subarc A_b of A that does not contain any inner point of an edge in E' for an endpoint b' of e_b .

By (4) we can find a spanning tree packing \mathcal{T} of $G - u$ such that there are two trees in \mathcal{T} whose union contains A_a , A_b and the edges $a'w$ and $b'w$. Clearly the union of the two paths and the two edges constitutes an a - b -bypass of v completing the proof of case 2a.

Next consider case 2b. Since there is no a - b -bypass of v in $H - u$ every such bypass in $G - u$ used w . Thus such a bypass in $G - u$ induces an a - a' -bypass and a b - b' -bypass of v in $H - u$ where a' and b' are vertices in V' or inner points of edges in E' .

We claim that in this case

(5) there is a spanning tree packing $\mathcal{T}_H = \{T_H^1, T_H^2, \dots, T_H^k\}$ of H such that $T_H^1 \cup T_H^2$ contains an a - a' -bypass and a b - b' -bypass of v .

By the induction hypothesis we can find a spanning tree packing $\mathcal{T}_a = \{T_a^1, T_a^2, \dots, T_a^k\}$ of $H - u$ such that $T_a^1 \cup T_a^2$ contains an a - a' -bypass of v . We can also find a spanning tree packing $\mathcal{T}_b = \{T_b^1, T_b^2, \dots, T_b^k\}$ of $H - u$ such that $T_b^1 \cup T_b^2$ contains a b - b' -bypass of v .

Since there is no a - b -bypass of v in $H - u$ we know that v is a cut vertex of $H - u$ and that a and b lie in different components of $H - \{u, v\}$. Denote by C_a the set of vertices of the component in which a lies and let $C_b = V_H \setminus (C_a \cup \{u, v\})$. It is easy to see that $T_a^i \{C_a\}$ —that is, the minor induced by C_a in T_a^i —is a spanning tree of $(H - u) \{C_a\}$ and that $T_b^i \{C_b\}$ is a spanning tree of $(H - u) \{C_b\} = (H - u)[V_H \setminus C_a]$. The statement (5) follows from Proposition 3.1.

Now let \mathcal{T}_H be a spanning tree packing of H as claimed in (5) and let A be an a - a' -bypass of v in $T_H^1 \cup T_H^2$. Define a vertex $a'' \in A$ and an a - a'' -subarc A' of A as follows.

If A contains an inner point of an edge in E' let e_a be the first edge in E' that we pass through when we traverse A starting at a . Let a'' be an endpoint of that edge such that A contains an a - a'' subarc A' that does not use inner points of any edge in E' . In this case an $a''w$ -edge results from pinching $e_a \in T_H^1 \cup T_H^2$

If A doesn't contain any inner point of an edge in E' then $a' \in V'$. Let $a'' = a'$ and $A' = A$. In this case there is an $a''w$ -edge which did not result from pinching any edge, in particular not from pinching an edge in T^i for $i > 2$.

Analogously define b'' and B' from a b - b' -bypass B of v .

Now we can by (4) find a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G - u$ such that $T^1 \cup T^2$ contains A' , B' and the edges $a''w$ and $b''w$. Clearly the union of the two paths and the two edges constitutes an a - b -bypass of w in $T^1 \cup T^2$. This completes the proof in case 2b.

Next let us turn to case 2c. In this case again the a - b -bypass of v in $G - u$ becomes an a - a' -bypass in $H - u$ where a' is a vertex in V' or an inner point of an edge in E' . By the induction hypothesis we can find a spanning tree packing $\mathcal{T}_H = \{T_H^1, T_H^2, \dots, T_H^k\}$ of $H - u$ such that $T_H^1 \cup T_H^2$ contains an a - a' -bypass A of v . Analogously to case 2b define a'' and an a - a'' -subarc A' of A .

Recall that in case 2c the point b is an inner point of an edge that has been split off to create an edge $e' \in E'$. We claim that

(6) we can choose \mathcal{T}_H in a way that $e' \in T_H^1 \cup T_H^2$.

If a and e' lie in the same component of $H - \{u, v\}$ we may assume that a' has been an inner point of e' in the first place and thus $e' \in T_H^1 \cup T_H^2$ holds.

So assume that a and e' lie in different components. In this case we can choose a vertex c in the component in which e' lies. Since there is a c - b' -bypass of v in $H - u$ for every inner point b' of e' we can apply the induction hypothesis and (5) to find a spanning tree packing with the desired properties.

This proves (6) and thus we can assume that the edge $b''w$ of which b is an inner point did not result from pinching an edge in T_H^i for $i > 2$.

By (4) we can find a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G - u$ such that $T^1 \cup T^2$ contains A' and the edges $a''w$ and $b''w$ whose union contains an a - b -bypass of v which completes the proof of case 2c.

In case 2d once again apply the induction hypothesis to find a spanning tree packing $\mathcal{T}_H = \{T_H^1, T_H^2, \dots, T_H^k\}$ of $H - u$ such that $T_H^1 \cup T_H^2$ contains an a - a' -bypass A of v where a' is a vertex in V' or an inner point of an edge in E' . As in the previous two cases define a'' and an a - a'' -subarc A' of A .

Now we can apply (4) to find a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G - u$ such that $T^1 \cup T^2$ contains A' and the edges $a''w$ and bw which proves (*) in case 2d.

In case 2e apply the induction hypothesis to find k edge disjoint spanning trees of $H - u$. If the edges $a'w$ and $b'w$ on which a and b lie were created by pinching some edges in E' we may permute the trees such that these edges lie in $T^1 \cup T^2$ or in none of the trees at all. We then apply (4) to find a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G - u$ such that $T^1 \cup T^2$ contains both $a'w$ and $b'w$. This proves that (*) holds in case 2e.

Since there are no more cases left it also completes the induction step and thus the proof of Lemma 4.2. \square

5. Results for Locally Finite Graphs

In the following sections we will prove Theorems 1.4 and 1.5. First let us recall the statements and give a brief outline of the proofs.

Theorem 1.4. *Let G be a $2k$ -edge connected locally finite graph with at most countably many ends. Then G admits an end faithful spanning tree packing of cardinality $k - 1$.*

Theorem 1.5. *Let G be a $2k$ -edge connected locally finite graph with at most countably many ends. Then G admits a topological spanning tree packing \mathcal{T} of cardinality k such that $T_1 \cup T_2$ is an end faithful connected spanning subgraph of G whenever $T_1, T_2 \in \mathcal{T}$ and $T_1 \neq T_2$.*

The graphs in the condition of the two theorems may be infinite in two ways: they are infinite in depth, i.e., there are rays and thus also ends, and they are infinite in width, i.e., there are infinitely many ends. In the proofs we will tackle these two types of infiniteness one by one. First we show the statements for one ended graphs by decomposing them into (countably many) finite graphs. Afterwards—using the results from the first part—we decompose graphs with countably many ends into finite and one ended graphs. Now let us give a brief summary of the proof steps.

As mentioned above we first consider the case where G has only one end. We choose a strictly increasing sequence G_n of finite contractions converging to G . The decomposition mentioned above consists of the graphs $G_n - G_{n-1}$ which are clearly finite. We inductively construct a sequence of spanning tree packings \mathcal{T}_n of G_n by choosing a spanning tree packing of $G_n - G_{n-1}$ (here Theorem 1.6 is essential) and combining this packing with the spanning tree packing \mathcal{T}_{n-1} . Taking the limit of the sequence \mathcal{T}_n and showing that it is an end faithful spanning tree packing of G finishes the proof in the one ended case.

If G has countably many ends we also select an increasing sequence G_n of contractions converging to G . This time however all of the contractions are infinite and G_{n+1} has exactly one end more than G_n , more precisely $G_{n+1} - G_n$ consists of a one ended and possibly a finite component. We apply the first part of the proof to inductively construct a sequence \mathcal{T}_n of end faithful spanning tree packings of G_n . Like in the one ended case the limit of the sequence \mathcal{T}_n is the desired end faithful spanning tree packing.

Clearly the crux in both parts of the proof is finding conditions under which the sequences \mathcal{T}_n converge to an end faithful spanning tree packing and constructing the \mathcal{T}_n accordingly. So in each of the two proof steps we establish such a condition before the actual proof.

In the second part of the proof it will also be important to choose the contractions in a way that we can apply the result of the first part. Hence before

moving to the proofs of Theorems 1.4 and 1.5 we show a result concerning cuts which is then used to define the contractions.

Now let us turn to the one ended case. We will prove the following statements for one ended graphs which are slightly stronger than Theorems 1.4 and 1.5. The stronger assertions are of vital importance for using the results in the proofs for graphs with countably many ends.

Theorem 5.1. *Let $G = (V, E)$ be a $2k$ -edge connected locally finite graph with only one end ω and let $V' \subseteq V$ such that the set of edges connecting V' to $V \setminus V'$ form a ω - V' -cut of minimal cardinality. Then $G - V'$ admits an end faithful spanning tree packing of cardinality $k - 1$.*

Theorem 5.2. *Let $G = (V, E)$ be a $2k$ -edge connected locally finite graph with only one end ω and let $V' \subseteq V$ such that the set of edges connecting V' to $V \setminus V'$ form a ω - V' -cut of minimal cardinality. Then $G - V'$ admits a topological spanning tree packing \mathcal{T} of cardinality k such that $T_1 \cup T_2$ is an end faithful connected spanning subgraph of G whenever $T_1, T_2 \in \mathcal{T}$ and $T_1 \neq T_2$.*

As mentioned earlier, before proving Theorems 5.1 and 5.2 we need a condition which ensures that the limit of a sequence of spanning trees of finite contractions is an end faithful spanning tree. Since the limit is a topological spanning tree by Lemma 3.3—note that in the finite case the notions of spanning trees and topological spanning trees coincide—we only need to ensure that it is connected.

Throughout the construction process we will take a look at potential pairs of components which will be formalised by the notion of gaps.

In the proof of Theorem 5.1 we would like to avoid that such a pair of potential components can propagate throughout the construction process because in this case we get at least two components in the limit.

In Theorem 5.2 the situation is slightly different. We allow the topological spanning trees to have more than one component. However, we require that the union with any other topological spanning tree in the limit is connected and end faithful. This will be achieved by bridging gaps, i.e., by constructing in each step paths in the union of two trees which connect the two potential components involved in a gap.

The following definition puts the intuitive notions above in a more formal context.

Definition 5.3. Let $G = (V, E)$ be a locally finite graph and $(V_n)_{n \in \mathbb{N}}$ an exhausting sequence of subsets of V . Define $G_n = G \setminus V_n$, i.e., G_n is the graph obtained from G by contracting each component of $G - V_n$ to a single vertex.

- Given a spanning tree T_n of G_n we call a pair of components of $T_n[V_n]$ a *gap* of T_n in G_n . If (C_1, C_2) is a gap and $u \in C_1$ and $v \in C_2$ we say that the gap *separates* u and v .
- Assume that for every $k \in \mathbb{N}$ we have a spanning tree T_k of G_k and that $T_{k+1}|_{G_k} = T_k$. Let $m < n$ be natural numbers.

A gap (C_1, C_2) of T_n *extends* a gap (C'_1, C'_2) of T_m if the vertex set of C'_i is a subset of the vertex set of C_i for $i \in \{1, 2\}$. Note that a gap (C'_1, C'_2) of T_m cannot be extended by multiple gaps in T_n . Hence we will not distinguish between a gap and its extension. This allows to talk about gaps in the sequence T_k .

A gap (C_1, C_2) of T_m *terminates* in V_n if it is not extended by a gap of T_n . Note that in this case C_1 and C_2 both are contained in the same component of $T_n[V_n]$ and thus connected by a path in T_n that does not use any contracted vertex. A gap that does not terminate is called a *persistent gap* in the sequence T_k .

- Given a gap $\Gamma = (C_1, C_2)$ of T_n we say that a tree $T'_n \subseteq G_n$ *bridges* the gap Γ in V_n if there is a path P from C_1 to C_2 in $T_n \cup T'_n$ which does not use any contracted vertex. The set of edges in $P \cap T'_n$ is called a Γ -*bridge* in T'_n .

A sequence T'_n bridges a gap Γ in a sequence T_n infinitely often if there are arbitrarily large sets of disjoint Γ -bridges in T'_n as $n \rightarrow \infty$. Note that this can only be the case if Γ is a persistent gap.

The following results verify the intuition that in order to obtain end faithful connected spanning subgraphs we only have to avoid persistent gaps. Note that they do not only hold for one-ended graphs but for arbitrary locally finite graphs, even those with uncountably many ends.

Lemma 5.4. *Let $G = (V, E)$ be a locally finite graph and $(V_n)_{n \in \mathbb{N}}$ an exhausting sequence of finite subsets of V . Furthermore let T_n, T'_n be sequences of spanning trees of $G[V_n]$ such that $T_n|_{V_{n-1}} = T_{n-1}$ and $T'_n|_{V_{n-1}} = T'_{n-1}$. If the sequence T'_n bridges every persistent gap of the sequence T_n infinitely often then $H := T \cup T'$ is an end faithful connected spanning subgraph of G where $T = \lim_{n \rightarrow \infty} T_n$ and $T' = \lim_{n \rightarrow \infty} T'_n$.*

Proof. The graph H is connected because any pair of components of H would constitute a persistent gap in the sequence T_n that is not bridged.

So we only need to show that any two rays γ_1 and γ_2 in H belonging to the same end ω of G are equivalent in H . Assume that γ_i belongs to an end ω_i of H for $i \in \{1, 2\}$ and that $\omega_1 \neq \omega_2$.

Let n be big enough that ω_1 and ω_2 lie in different components C^1_H and C^2_H of $H - V_n$. Note that C^1_H and C^2_H both are subsets of the same component C_G of $G - V_n$ because γ_1 and γ_2 converge to the same end of G . For $i \in \{1, 2\}$ denote by $v_i \in C^i_H$ a vertex of γ_i such that all consecutive vertices lie in C^i_H as well.

If v_1 and v_2 belonged to the same component of T then the unique path in T connecting v_1 and v_2 would use vertices in V_n . Hence this path would correspond to a cycle in T_n . So v_1 and v_2 belong to different components of T and there is a persistent gap Γ that separates v_1 and v_2 .

We know that Γ is bridged infinitely often, i.e., there are arbitrarily large sets \mathcal{P} of paths in H connecting v_1 and v_2 such that

$$\forall P_1, P_2 \in \mathcal{P}: P_1 \cap P_2 \cap E(T') = \emptyset.$$

If a path $P \in \mathcal{P}$ has non-empty intersection with V_n it has to use at least one edge of T' with one endpoint in V_n because otherwise $P|_{G_n}$ would be a cycle in T_n . Hence the number of such paths is bounded by the number of edges with an endpoint in V_n . So if $|\mathcal{P}|$ is large enough then there is a path $P^* \in \mathcal{P}$ that connects v_1 and v_2 such that P^* contains no vertex of V_n .

Consequently v_1 and v_2 lie in the same component of $H - V_n$, a contradiction. \square

Lemma 5.5. *Let $G = (V, E)$ be a locally finite graph and $(V_n)_{n \in \mathbb{N}}$ an exhausting sequence of finite subsets of V . Furthermore let T_n be a sequence of spanning trees of $G\{V_n\}$ such that $T_n|_{V_{n-1}} = T_{n-1}$. If the sequence T_n contains no persistent gaps then $T := \lim_{n \rightarrow \infty} T_n$ is an end faithful spanning tree of G .*

Proof. Since T_n contains no persistent gaps it is clear that T is end faithful and connected by Lemma 5.4 with $T'_n = T_n$. We also know that T is a topological spanning tree and thus acyclic—note that finite circles constitute a special case of topological circles. Hence T is both a spanning tree and a topological spanning tree and Theorem 3.4 completes the proof \square

With these auxiliary results on hand we can finally move on to the proofs of Theorems 5.1 and 5.2. In both of the proofs we will construct a sequence of finite contractions and corresponding spanning tree packings fulfilling the condition of Lemmas 5.4 and 5.5 respectively. This will be done in detail for Theorem 5.2 while for Theorem 5.1 we will only indicate the modifications to be made.

Proof of Theorem 5.2. Denote by ω the unique end of G and let $V_0 := V'$. It is easy to see that the following construction of an exhausting sequence of sets of vertices is always possible.

For every $n \in \mathbb{N}$ choose a finite set $W_n \supseteq V_{n-1}$ of vertices containing all neighbours of V_{n-1} such that $G[W_n \setminus V_{n-1}]$ is connected and that there is only one component of $G - W_n$. Let S_n be a W_n - ω -cut of minimal cardinality and define V_n to be the component of $G - S_n$ in which W_n lies.

Next we construct a sequence $\mathcal{T}_n = \{T_n^1, T_n^2, \dots, T_n^k\}$ of spanning tree packings of $G_n := G\{V_n\} - V_0$ such that

- (1) $T_{n+1}^i|_{G_n} = T_n^i$ and
- (2) for $i \neq j$ every persistent gap in T_n^i is bridged infinitely often by T_n^j .

Then $T^i = \lim_{n \rightarrow \infty} T_n^i$ is a topological spanning tree of $G - V'$ by Lemma 3.3. Whenever $i \neq j$ the edge sets of T^i and T^j are disjoint because T_n^i and T_n^j are edge disjoint for every $n \in \mathbb{N}$ and by Lemma 5.4 the graph $T^i \cup T^j$ is an end faithful connected spanning subgraph of G .

For the construction of the spanning tree packings note that the graph $G'_n := G\{V_n \setminus V_{n-1}\}$ is $2k$ -edge connected. Let u_n be the contracted vertex in G'_n that corresponds to V_{n-1} and denote by v_n the only other contracted vertex in G'_n . Then the set S_n of edges incident to u_n is a u_n - v_n -cut of minimal cardinality. This implies that we can apply Theorem 1.6 to each of the G'_n .

Start by selecting an arbitrary spanning tree packing $\mathcal{T}_1 = \{T_1^1, T_1^2, \dots, T_1^k\}$ of G_1 . Such a spanning tree packing exists by Theorem 1.6.

For $n \geq 2$ we will inductively define \mathcal{T}_n from \mathcal{T}_{n-1} by selecting a spanning tree packing \mathcal{T}'_n of $G'_n - u_n$. Consequently Proposition 3.1 can be applied to find a spanning tree packing of G_n for which (1) holds.

In order to take care of (2) we will bridge one possible gap in every step of the construction process. For this purpose let w_1, w_2, w_3, \dots be an enumeration of the vertices of G and choose a function

$$\varphi: \mathbb{N} \rightarrow \mathbb{N}^2 \times \{(i, j) \mid 1 \leq i, j \leq k \text{ and } i \neq j\}$$

such that $\varphi^{-1}(s)$ is infinite for every s . This function will be used to make sure that every persistent gap is bridged infinitely often in the following way: if $\varphi(n) = (n_1, n_2, i, j)$ and there is a gap Γ in T_{n-1}^i that separates w_{n_1} and w_{n_2} we construct a Γ -bridge in $T_n^j - V_{n-1}$. So if the gap Γ is persistent there will be arbitrarily large sets of disjoint Γ -bridges since $\varphi^{-1}(n_1, n_2, i, j)$ is infinite.

All that is left to show now is that we can construct such a Γ -bridge, i.e., that we can choose the spanning tree packing of $G'_n - u_n$ accordingly. For this purpose let P be the unique path in T_{n-1}^i that connects w_{n_1} and w_{n_2} . Note that we may assume that $w_{n_1}, w_{n_2} \in V_{n-1}$ and that P uses the unique contracted vertex in G_{n-1} . Otherwise no gap Γ separates w_{n_1} and w_{n_2} and thus there is nothing to show.

Denote by a_n and b_n the vertices in G'_n that are incident to edges used by P such that $a_n, b_n \neq u_n$. Both a_n and b_n are contained in W_n and since $G[W_n \setminus V_{n-1}]$ is connected, there is an a_n - b_n -bypass of v_n in $G'_n - u_n$. So Theorem 1.6 can be used to find a spanning tree packing $\mathcal{T}'_n = \{T'_{n,1}, \dots, T'_{n,k}\}$ of $G'_n - u_n$ such that $T'_{n,i} \cup T'_{n,j}$ contains an a_n - b_n -bypass P_n of v_n .

It is immediate that the union $P^* = P \cup P_n$ is a w_{n_1} - w_{n_2} -path in $T_n^i \cup T_n^j$ that does not use any contracted vertex. So P^* contains a Γ -bridge in T_n^j . All edges of this bridge are contained in $T_n^j - V_{n-1}$ because $P^* \setminus G\{V_{n-1}\}$ is a path in T_{n-1}^i . \square

Proof Sketch of Theorem 5.1. Construct a sequence of spanning tree packings as we did in the proof of Theorem 5.2.

This time however, instead of Theorem 1.6 we will use Corollary 4.3 to obtain a_n - b_n -paths that are contained in T_n^i and consequently the v_{n_1} - v_{n_2} -path P^* is completely contained in T_n^i . So if at some point in the construction process there is a gap separating v_{n_1} and v_{n_2} it will eventually terminate.

Now use Lemma 5.5 to conclude that the limit of the spanning tree packings is an end faithful spanning tree packing of G . \square

This completes the proof in the one-ended case. In the case of graphs with countably many ends we will again define a sequence G_n of contractions of G converging to G and a sequence of spanning tree packings \mathcal{T}_n of G_n converging to the desired spanning tree packings.

Since we would like to use Theorems 1.6, 5.1 and 5.2 for the construction of the \mathcal{T}_n we need to define the contractions in a way that this is possible. To motivate the definition given below consider for a moment the following situation: we have constructed a contraction G_{n-1} and an end faithful spanning tree packing of this contraction. Now we wish to decontract a contracted vertex v and use one of the theorems in the subgraph induced by the corresponding vertex set. Clearly it is essential that the cut formed by the edges incident to the contracted vertex is a cut of minimal cardinality in the minor induced by the set of vertices corresponding to v .

We have no sufficient means of controlling the order of the vertices being decontracted during the proof. Hence the above should be true for each contracted vertex at any time. This implies that we need cuts of minimal cardinality which do not interfere with each other.

Definition 5.6. Let $G = (V, E)$ be a graph, let $x \in V \cup \Omega$ and let $Y \subseteq (V \cup \Omega) \setminus \{x\}$. Let S_y be a x - y -cut for $y \in Y$ and denote by C_y the component of $G \setminus S_y$ in which y lies. The set $\{S_y \mid y \in Y\}$ is said to be *compatible* if for any $y, y' \in Y$ the cut S_y does not contain an edge that connects two points in $C_{y'}$, otherwise it is said to be *incompatible*. If the set $\{S_1, S_2\}$ is compatible we will call the cuts S_1 and S_2 compatible.

Clearly a compatible set of cuts of minimal cardinality is what is needed. The following definition can be used to provide a way of constructing such a set.

Definition 5.7. Let $G = (V, E)$ be a graph, $x \in V \cup \Omega$ and let $Y \subseteq (V \cup \Omega) \setminus \{x\}$. Define

$$\mathfrak{C}_x^Y = \{S \mid S \text{ is a } x\text{-}y\text{-cut of minimal cardinality for some } y \in Y\}$$

and denote by \mathfrak{S}_x^Y the power set of \mathfrak{C}_x^Y , i.e., the elements of \mathfrak{S}_x^Y are sets of cuts.

Now we define a binary relation \sqsubset on $\mathfrak{S}_x^Y \times \mathfrak{S}_x^Y$. We say that $\mathcal{S} \sqsubset \mathcal{U}$ if

- (D1) $|\mathcal{S}| \leq |\mathcal{U}|$,
- (D2) $S \subseteq \bigcup_{U \in \mathcal{U}} U$ for every $S \in \mathcal{S}$ and
- (D3) the component of $G \setminus \bigcup_{S \in \mathcal{S}} S$ in which x lies is exactly the component of $G \setminus \bigcup_{U \in \mathcal{U}} U$ in which x lies.

Remark. Clearly $\mathcal{S} \sqsubset \mathcal{U}$ and $\mathcal{U} \sqsubset \mathcal{S}$ implies that $\bigcup_{S \in \mathcal{S}} S = \bigcup_{U \in \mathcal{U}} U$. It is also an easily observed fact that the relation \sqsubset is transitive and that whenever $U \in \mathfrak{C}_x^Y$ and $U \notin \mathcal{U}$ then $\mathcal{S} \sqsubset \mathcal{U}$ implies that $\mathcal{S} \cup \{U\} \sqsubset \mathcal{U} \cup \{U\}$.

Now starting with an arbitrary finite set \mathcal{U} of minimal cuts we can find a compatible set \mathcal{S} of minimal cuts which satisfies $\mathcal{S} \sqsubset \mathcal{U}$.

Lemma 5.8. *Let $G = (V, E)$ be a locally finite graph, $x \in V \cup \Omega$ and let $Y \subseteq (V \cup \Omega) \setminus \{x\}$. Let $\mathcal{U} \in \mathfrak{S}_x^Y$ be finite. Then there is a compatible set $\mathcal{S} \in \mathfrak{S}_x^Y$ such that $\mathcal{S} \sqsubset \mathcal{U}$.*

Proof. We will prove Lemma 5.8 by induction on $|\mathcal{U}|$. For $|\mathcal{U}| = 1$ there is nothing to show because a set of one cut is always compatible.

For $|\mathcal{U}| > 1$ let $U \in \mathcal{U}$ and apply the induction hypothesis to $\mathcal{U} \setminus \{U\}$ to obtain a compatible set $\mathcal{S}' \sqsubset \mathcal{U} \setminus \{U\}$ of cuts. By the above remark $\mathcal{S}' \cup \{U\} \sqsubset \mathcal{U}$. Clearly $|\mathcal{S}'| \leq |\mathcal{U}| - 1$.

If the inequality is strict we can apply the induction hypothesis again to $\mathcal{S}' \cup \{U\}$ to obtain $\mathcal{S} \sqsubset \mathcal{S}' \cup \{U\}$ where \mathcal{S} is compatible. From the above remark it follows that $\mathcal{S} \sqsubset \mathcal{U}$ which completes the proof.

So assume that $|\mathcal{S}'| = |\mathcal{U}| - 1$. Choose a cut $S_1 \in \mathfrak{e}_x^Y$ fulfilling $\mathcal{S}' \cup \{S_1\} \sqsubset \mathcal{U}$ with the property that the number of cuts $S' \in \mathcal{S}'$ that are incompatible with S_1 is minimal. Note that there is such a cut because $\mathcal{S}' \cup \{U\} \sqsubset \mathcal{U}$.

If S_1 is compatible with all cuts in \mathcal{S}' we are done. So assume that there is a cut $S_2 \in \mathcal{S}'$ such that S_1 and S_2 are incompatible. For $i \in \{1, 2\}$ let $y_i \in Y$ be such that S_i is a x - y_i -cut of minimal cardinality.

For the next step of the proof we will need some definitions. Denote by C_0 the component of $G \setminus (S_1 \cup S_2)$ in which x lies. Let C_i be the component of $G \setminus S_i$ in which y_i lies. Let A_1 be the set of edges connecting $C_1 \setminus C_2$ to C_0 and let B_1 be the set of edges connecting $C_1 \cap C_2$ to $C_2 \setminus C_1$. Analogously define A_2 and B_2 . Let C be the set of edges between $C_1 \setminus C_2$ and $C_2 \setminus C_1$ and let D be the set of edges connecting C_0 and $C_1 \cap C_2$.

From the definitions it is clear that A_1, A_2, B_1, B_2, C and D are pairwise disjoint and that $S_i = A_i \cup B_i \cup C \cup D$. It is also clear that for the properties (D1) to (D3) it does not matter if an edge of B_1, B_2 or C is contained in any cut.

We will now define a new cut which—depending on where y_1 and y_2 lie—can either be used to replace S_1 and S_2 so that we can apply the induction hypothesis again or contradicts the assumption that the number of cuts $S' \in \mathcal{S}'$ that are incompatible with S_1 is minimal.

- Suppose that y_1 is contained in $C_1 \cap C_2$. Since S_i is a x - y_i -cut of minimal cardinality it follows that

$$\begin{aligned} |A_1| + |B_1| + |C| + |D| \leq |B_1| + |B_2| + |D| &\Rightarrow |A_1| + |C| \leq |B_2|, \\ |A_2| + |B_2| + |C| + |D| \leq |A_1| + |A_2| + |D| &\Rightarrow |B_2| + |C| \leq |A_1|. \end{aligned}$$

This implies that $|C| = 0$ and $|A_1| = |B_2|$. Hence

$$|A_2| + |B_2| + |C| + |D| = |A_1| + |A_2| + |D|$$

and thus the cut defined by $S^* := A_1 \cup A_2 \cup D$ is an x - y_2 -cut of minimal cardinality. It is easy to see that $(\mathcal{S}' \setminus S_2) \cup S^* \sqsubset \mathcal{U}$ and since $(\mathcal{S}' \setminus S_2) \cup S^*$ has strictly less elements than \mathcal{U} we can apply the induction hypothesis to find a compatible set $\mathcal{S} \sqsubset (\mathcal{S}' \setminus S_2) \cup S^* \sqsubset \mathcal{U}$.

- For $y_2 \in C_1 \cap C_2$ an analogous argument to the previous case works.
- Finally assume that $y_1 \in C_1 \setminus C_2$ and $y_2 \in C_2 \setminus C_1$. Then

$$\begin{aligned} |A_1| + |B_1| + |C| + |D| \leq |A_1| + |B_2| + |C| &\Rightarrow |B_1| + |D| \leq |B_2|, \\ |A_2| + |B_2| + |C| + |D| \leq |A_2| + |B_1| + |C| &\Rightarrow |B_2| + |D| \leq |B_1|. \end{aligned}$$

This in particular implies that $|D| = 0$ and $|B_1| = |B_2|$ and thus

$$|A_1| + |B_1| + |C| + |D| = |A_1| + |B_2| + |C|.$$

So the cut $S_1^* := A_1 \cup B_2 \cup C$ is an x - y_1 -cut of minimal cardinality.

Clearly $\mathcal{S}' \cup \{S_1^*\} \sqsubset \mathcal{S}' \cup \{S_1\} \sqsubset \mathcal{U}$. Whenever S_1 and $S' \in \mathcal{S}'$ are compatible it can easily be seen that S_1^* and S' are compatible as well (recall that S' and S_2 are compatible). The cuts S_1^* and S_2 are compatible while S_1 and S_2 are incompatible. So the number of cuts in \mathcal{S}' that are incompatible to S_1^* is strictly smaller than the number of cuts that are incompatible to S_1 . This contradicts S_1 minimizing that number. \square

Another tool needed in the construction of the contractions are rays that completely cover an end in the following sense.

Definition 5.9. A ray γ is said to *devour* an end ω if every ω -ray meets γ .

From the existence of normal spanning trees (cf. [14, Theorem 8.2.4]) one can easily derive that such rays always exist.

Lemma 5.10. *Let $G = (V, E)$ be a locally finite graph, $v \in V$, and let ω be an end of G . Then there is a ray γ in G that starts in v and devours ω .*

Proof. Consider a normal spanning tree with root v . The normal ω -ray in this tree has the desired property. \square

Now we have all means for constructing the sequence of contractions. The only thing that is still required for proving Theorems 1.4 and 1.5 are conditions to ensure that the topological spanning trees which we get in the limit have the desired properties. The next two results provide us with such conditions.

Lemma 5.11. *Let G be a locally finite graph and let T be a topological spanning tree of G such that any two topological rays of T that converge to the same end of G have a common topological tail. Then $T \cap G$ is an end faithful spanning tree of G .*

Proof. Since T contains no topological circle it doesn't contain a finite circle. So we only have to prove that $T \cap G$ is connected because in this case $T \cap G$ is a spanning tree that induces a topological spanning tree and thus it is end faithful by Theorem 3.4.

So assume that $T \cap G$ was not connected and let u and v be vertices that lie in different components of $T \cap G$. Since T is arcwise connected there is a

u - v -arc A in T which clearly has to contain some end ω . It is immediate that $A - \omega$ consists of two disjoint topological rays starting in u and v respectively. Both of those topological rays converge to the same end ω but they do not have a common topological tail, a contradiction. \square

Lemma 5.12. *Let G be a locally finite graph, let H be a subgraph of G and let $T \subseteq H$ be an end faithful spanning tree of G . Then H is end faithful.*

Proof. Let ω be an end of G . Then there is a ray γ in T that belongs to ω since T is end faithful. Hence H contains a ray to every end of G . So we only need to show that every ray γ' in H that belongs to ω is equivalent to γ in H . By the Star-Comb-Lemma 2.3 the tree T contains an infinite comb with all teeth in γ' (note that T is locally finite). The spine of this comb lies in ω and thus is either a tail of γ or vice versa. \square

Now we are finally in a position to prove Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Let $(v_n)_{n \in \mathbb{N}}$ and $(\omega_n)_{n \in \mathbb{N}}$ be enumeration of all vertices and ends of $G = (V, E)$ respectively. We will now define contractions G_n of G such that $v_k \in G_n$ for all $k \leq n$ and the ends of G_n are exactly $\{\omega_1, \dots, \omega_n\}$. Furthermore we require all of the G_n to be locally finite.

Then we construct end faithful spanning tree packings $\mathcal{T}_n = \{T_n^1, \dots, T_n^{k-1}\}$ of G_n such that $T_n^i|_{G_{n-1}} = T_{n-1}^i$. By Lemma 3.3 the limit of this sequence is a topological spanning tree packing of G and we will apply Lemma 5.11 to show that it is indeed an end faithful spanning tree packing.

For the construction of the contractions G_n let G_0 be the graph on one vertex, i.e., all of V has been contracted. The graph G_{n-1} is locally finite and does not contain an ω_n -ray. Hence there must be a vertex v in G_{n-1} which resulted from contracting a subgraph H_n of G containing a tail of every ω_n -ray.

Select a ray γ in H_n that devours ω_n , i.e., $H_n - \gamma$ doesn't contain an ω_n -ray. Such a ray exists by Lemma 5.10. For every $N > n$ such that a ray of ω_N lies in H_n select a cut S_N of minimal cardinality in H_n separating ω_N from the set of vertices of γ . Note that every such cut is finite because there are only finitely many edge disjoint rays in ω_N that start in γ (if there were infinitely many such rays then γ would lie in $\omega_N \neq \omega_n$, a contradiction).

Now consider the component C_{ω_n} of $H_n - \bigcup S_N$ in which γ lies. Clearly all rays in C_{ω_n} belong to ω_n because every other end has been separated from γ by one of the cuts S_N .

For each component C of $H_n - C_{\omega_n}$ the set of edges connecting C_{ω_n} to C is finite because otherwise we would by the Star-Comb-Lemma 2.3 get an ω_n -ray in C contradicting the assumption that γ devours ω_n . Hence we can choose a finite set $\mathcal{S}_C^{\omega_n}$ of cuts S_N containing all of these edges and by Lemma 5.8 we can find a compatible set of cuts $\tilde{\mathcal{S}}_C^{\omega_n} \sqsubset \mathcal{S}_C^{\omega_n}$.

Having chosen such compatible sets of cuts for each component of $H_n - C_{\omega_n}$ we contract every component of $H - \bigcup_C \bigcup_{S \in \tilde{\mathcal{S}}_C^{\omega_n}} S$ except C_{ω_n} to a vertex and denote the resulting graph by H'_n .

Define G'_n to be the graph obtained from G_{n-1} by replacing v by H'_n . Clearly G'_n is a contraction of G with exactly the desired ends. If $v_n \in G'_n$ we choose $G_n = G'_n$. Otherwise decontract the contracted vertex w of G'_n that contains v_n and denote the part of the resulting graph that corresponds to w by K_n , i.e., K_n is an induced subgraph of G defined similar as H_n above.

For every end ω of K_n choose a ω - v_n -cut of minimal cardinality and denote the set of these cuts by \mathcal{S} . The component C_{v_n} of $G - \bigcup_{S \in \mathcal{S}} S$ in which v_n lies is obviously finite. Hence we can find a finite subset $\mathcal{S}^{v_n} \subseteq \mathcal{S}$ such that the component of $G - \bigcup_{S \in \mathcal{S}^{v_n}} S$ in which v_n lies is exactly C_{v_n} . Now apply Lemma 5.8 to find a compatible set $\tilde{\mathcal{S}}^{v_n} \sqsubset \mathcal{S}^{v_n}$.

Finally contract every component of $K_n - \bigcup_{S \in \tilde{\mathcal{S}}^{v_n}} S$ except for the one containing v_n to a vertex and denote the resulting graph by K'_n . We obtain G_n from G'_n by replacing w by K'_n .

Clearly G_n is a contraction of G . It is locally finite because

- G_{n-1} is locally finite,
- G is locally finite, so all vertices of H'_n and K'_n that are not contracted have finite degree, and
- all newly contracted vertices have finite degree since the cuts that separate the corresponding vertex sets from the rest of the graph are finite.

The next step of the proof is to construct an end faithful spanning tree packing \mathcal{T}_n of cardinality $k-1$ of G_n out of such a packing \mathcal{T}_{n-1} of G_{n-1} . Proposition 3.1 and Theorem 3.4 imply that it is sufficient to find $k-1$ edge disjoint spanning trees of K'_n and $k-1$ edge disjoint end faithful spanning trees of H'_n .

Consider the graph H''_n obtained from G_n by contracting $G_n - H'_n$ to a single vertex x (so $H'_n = H''_n - x$). By the construction of the sets $\tilde{\mathcal{S}}_C^{\omega_n}$ and $\tilde{\mathcal{S}}^{v_n}$ of cuts we know that the set U of edges connecting H'_n to $G_n - H'_n$ is a cut of minimal cardinality separating $G_n - H'_n$ and some end ω of H_n . Hence U is either an ω_n - x -cut of minimal cardinality or an x - y -cut of minimal cardinality for some contracted vertex y of H'_n .

In the first case we can apply Theorem 5.1 directly to obtain an end faithful spanning tree packing of H'_n .

In the latter case choose an ω_n - x -cut U' of minimal cardinality in H''_n . Clearly the graph $H''_n - U'$ has two components one of which is infinite. Denote the vertex set of the finite component by C_∞ and the vertex set of the infinite component by C_ω .

Now U is an x - y -cut of minimal cardinality for some contracted vertex y of H'_n $H''_n \setminus \{C_\omega\}$. Thus we can apply Corollary 4.3 to find a spanning tree packing of cardinality $k-1$ of $H''_n \setminus \{C_\omega\} - x$. Furthermore by Theorem 5.1 we can find $k-1$ edge disjoint end faithful spanning trees of $H''_n \setminus \{C_\omega\}$. Proposition 3.1 yields the desired end faithful spanning tree packing of $H''_n - x = H'_n$.

The spanning tree packing of K'_n can be found by Corollary 4.3 for the same reasons as above.

Finally we need to show that the limit of the sequence of spanning tree packings that we just constructed is indeed an end faithful spanning tree packing. By Lemma 3.3 it is a topological spanning tree packing because every end faithful spanning tree is also a topological spanning tree. Hence to be able to apply Lemma 5.11 we only need to show that for each $T \in \lim_{n \rightarrow \infty} \mathcal{T}_n$ and for every end ω_n of G any two topological ω_n -rays in T have a common tail.

Assume that this was not the case and let γ_1 and γ_2 be topological ω_n -rays with no common tail. The restrictions γ'_1 and γ'_2 of the two rays to G_n are again topological ω_n -rays in $T_n = T|_{G_n} \in \mathcal{T}_n$ with no common tail.

Since G_n has only finitely many ends we may assume that γ'_1 and γ'_2 are rays (otherwise choose tails containing no ends). Now γ'_1 and γ'_2 are equivalent in G_n but not in T_n , a contradiction to T_n being an end faithful spanning tree of G_n . \square

The construction process in the proof of Theorem 1.5 is analogous to what was done in the previous proof hence we do not describe it in detail. Instead we indicate the modifications to be made and show how Lemma 5.12 can be used to show that the topological spanning tree packing has the desired properties.

Proof Sketch of Theorem 1.5. For the construction of the the topological spanning tree packing follow the construction in the proof of Theorem 1.4 (use Theorem 1.6 instead of Corollary 4.3 and Theorem 5.2 instead of Theorem 5.1).

To see that the union of two trees $T^i = \lim T_n^i$ and $T^j = \lim T_n^j$ is end faithful we construct a sequence of end faithful spanning trees of $T_n^i \cup T_n^j$ (and thus also of G_n) in every step of the construction process. We can do so by choosing an end faithful spanning tree of the graph H'_n and a spanning tree of K'_n which are contained in $T_n^i \cup T_n^j$ and applying Proposition 3.1.

The limit of these spanning trees is an end faithful spanning tree by the same arguments as in the proof of Theorem 1.4 and so we can apply Lemma 5.12 to show that $T^i \cup T^j$ is indeed an end faithful spanning subgraph of G . \square

6. An Application: Hamiltonian Cycles in Locally Finite Line Graphs

As already mentioned in the introduction the results of the previous section can be used to partially verify Georgakopoulos' Conjecture 1.3. More precisely we can show that the following special case holds.

Theorem 1.7. *The line graph of every locally finite 6-edge connected graph with at most countably many ends has a Hamiltonian circle.*

This theorem extends a result by Brewster and Funk [2] who showed that Georgakopoulos' conjecture is true for 6-edge connected graphs with finitely many ends all of which are thin.

To prove Theorem 1.7 we will combine one of Theorems 1.4 and 1.5 with some known results from infinite graph theory. It is worth noting that the finite counterpart can be proved in a completely analogous way. Hence for every step we will also provide a brief sketch of the finite analogue. Basically the proof can be outlined as follows.

1. Find two edge disjoint topological spanning trees T, T' such that $T \cap G$ is an ordinary spanning tree of G .
2. Use the trees T and T' to construct an end faithful Eulerian subgraph H of G .
3. From an Euler tour in H construct a Hamilton cycle in $L(G)$.

In the finite case one simply needs to find two edge disjoint spanning trees in the first step. So Theorem 1.1 can be used. Either of Theorems 1.4 and 1.5 can be used in the infinite case. Note that for a finite graph it suffices if it is 4-edge connected while in the general case we require an edge connectivity of at least 6.

The second step uses the notion of fundamental circles where the fundamental circle of an edge e with respect to a (topological) spanning tree T is the unique (topological) circle in $T + e$. Now, in the finite case let H be the subgraph of G which contains an edge $e \in E$ if and only if it is contained in an odd number of fundamental circles of edges in T' with respect to T . Clearly H is connected because $T' \subseteq H$ and it is easily seen that all vertex degrees in H are even. Hence H is a spanning Eulerian subgraph of G .

In the infinite case we would like to use the same construction. To make this well-defined it is necessary that every edge is only contained in finitely many of the fundamental circles. Fortunately, by the following result this is always the case.

Theorem 6.1 (Diestel and Kühn [16]). *Let G be a locally finite graph and let T be a topological spanning tree of G . Then every edge is only contained in finitely many fundamental circles with respect to T .*

Next we need to define what we mean by an infinite Eulerian graph and to make sure that the construction used above really yields such a graph.

Definition 6.2. A *topological Euler tour* of a graph G is a continuous map $\sigma: S^1 \rightarrow \overline{G}$ such that every inner point of an edge is the image of exactly one point of S^1 . A locally finite graph is called *Eulerian* if it admits a topological Euler tour.

Above we used the well-known result from finite graph theory that a graph is Eulerian if and only if all vertex degrees are even. The following theorem by Diestel and Kühn provides a similar characterisation for infinite Eulerian graphs.

Theorem 6.3 (Diestel and Kühn [15]). *For a locally finite graph G the following two statements are equivalent.*

1. G is Eulerian.
2. Every cut of G is either even or infinite.

With this result on hand it can easily be seen that the construction from the finite case also yields an Eulerian subgraph in the locally finite case.

Proposition 6.4. *Let G be a locally finite graph and let T, T' be topological spanning trees of G . Let H be the subgraph of G which contains all edges of T and those edges of T' that are in an odd number of fundamental circles of edges in T . Then H contains an even number of edges in S for every finite cut S in G .*

Proof. Denote by K_S the set of fundamental circles that contain edges in S for some finite cut S and denote by 1_K the indicator function of the edge set of a given circle K . Clearly every such circle contains an even number of edges in S and thus

$$|S \cap H| \equiv \sum_{e \in S} \sum_{K \in K_S} 1_K(e) \equiv \sum_{K \in K_S} \sum_{e \in S} 1_K(e) \equiv \sum_{K \in K_S} |K \cap S| \equiv 0 \pmod{2}$$

because all the sums are finite by Theorem 6.1. \square

The following result finishes the second proof step in the infinite case.

Proposition 6.5. *Let G be a locally finite graph and let T and T' be two edge disjoint topological spanning trees of G such that $T \cap T'$ is an ordinary spanning tree of G . Then G has an end faithful spanning subgraph that admits a topological Euler tour.*

Proof. Let H be the graph of Proposition 6.4. Then H is connected because $T \subseteq H$ and it is end faithful by Lemma 5.12.

Furthermore H contains an even number of edges of every finite cut of G . Since T is an end faithful spanning tree it contains an infinite number of edges of every infinite cut of G and thus so does H . Hence every finite cut of H is even and so H is Eulerian. \square

In the third proof step for finite graphs we can use the following result by Harary and Nash-Williams and the fact that H is Eulerian.

Proposition 6.6 (Harary and Nash-Williams [22]). *$L(G)$ is Hamiltonian if and only if there is a tour in G which includes at least one endpoint of each edge.*

For locally finite graphs a similar result is known.

Definition 6.7. A *closed dominating trail* is a topological Euler tour of a connected subgraph H of G which contains at least one endvertex of each edge of G .

Proposition 6.8 (Brewster and Funk [2]). *Let $G = (V, E)$ be a locally finite graph. If \overline{G} contains a closed dominating trail which is injective on the ends of G then $L(G)$ is Hamiltonian.*

Note that the tour is required to be injective at ends. Even though the Eulerian subgraph that we constructed in the second proof step is end faithful a badly chosen Euler tour may visit an end more than once. However, the following result by Georgakopoulos can be used to find a topological Euler tour which is injective at the ends of H and thus (since H is end faithful) also at the ends of G .

Theorem 6.9 (Georgakopoulos [18]). *If a locally finite multigraph has a topological Euler tour, then it also has one that is injective at ends.*

Since a topological Euler tour in a spanning connected subgraph clearly is a dominating closed trail this completes the proof of Theorem 1.7.

7. The counterexample of Aharoni and Thomassen

We mentioned earlier that Theorems 1.4 and 1.5 do not remain true if the graph has uncountably many ends. The following construction due to Aharoni and Thomassen [1] shows that for every $k \in \mathbb{N}$ there is a k -edge connected locally finite graph G such that there is no edge disjoint pair consisting of a connected spanning subgraph of G and an arcwise connected subspace of \overline{G} containing all vertices of G . This clearly implies that the statements of Theorems 1.4 and 1.5 are not true for graphs with edge connectivity ≥ 6 and uncountably many ends because both of the theorems give us such a pair.

For the construction let $k \in \mathbb{N}$ be fixed and choose a finite k -edge connected graph H such that there is a set S of k vertices whose pairwise distance is at least k .

Let $G_1 = H$ and let $V_0 = \emptyset$. In order to define G_{n+1} start with G_n and let $G'_n = G_n - V_{n-1}$. Subdivide every edge e of G'_n by adding $p_k(e)$ inner vertices where $p_k(e)$ be the number of paths of length k in G'_n containing e . We obtain a subdivision of G_n whose vertex set is denoted V_n .

Now add a copy of H for every path of length k in G'_n such that all the copies are mutually disjoint. Identify each vertex of a copy of S with an inner point of a different edge in the corresponding path in G'_n . The identification is done in a way such that the copies of H remain disjoint.

Define a limit graph G on the vertex set $V = \bigcup_{n \geq 1} V_n$ by connecting a vertex $v \in V_n$ to all of its neighbours in G_{n+1} .

An inductive argument shows that G_n and hence also G is k -edge connected. It is also easy to see that the edge set of the subdivision of a path of length k in G'_n separates the corresponding copy of H from V_n and from all other copies of H added in step $n + 1$.

To see that there is no edge disjoint pair of a connected subgraph of G and an arcwise connected subspace of \overline{G} containing all vertices of G consider a path P connecting two vertices in the copy of S in V_1 . Every connected spanning subgraph of G must contain such a path. There is a maximal n such that this path intersects V_n . This implies that P contains the subdivision of some path of length k in G'_n and thus a finite cut.

This shows that Theorems 1.4 and 1.5 cannot be extended to graphs with uncountably many ends. However, it might still be true to extend Theorem 1.7 to such graphs.

Proposition 7.1. *Let G be a graph constructed according to the above procedure. Then $L(G)$ is Hamiltonian.*

Proof. While it is not possible to find a suitable spanning tree packing we can still find a spanning end faithful Eulerian subgraph of G . Hence $L(G)$ is Hamiltonian by Theorem 6.9 and Proposition 6.8.

Let H be the k -edge connected graph used in the construction. Since $k \geq 4$ we know that H contains a pair of edge disjoint spanning trees T_1 and T_2 .

It follows that H contains a connected spanning Eulerian subgraph. Simply take all edges which are contained in an odd number of fundamental cycles with respect to T_1 . It is easy to see that every vertex even degree in the resulting graph and it is connected because it contains all of T_2 . Observe that flipping all edges—that is, remove an edge if it was present, add it if it was not—of the unique u - v -path in T_1 gives a connected spanning subgraph of H which contains a Eulerian u - v -trail, that is, the only vertices with odd degree in the subgraph are u and v .

Now let K be a connected spanning Eulerian subgraph of H and let K_{uv} be a connected spanning subgraph containing a Eulerian u - v -trail. We will use these graphs to iteratively define connected spanning Eulerian subgraphs L_n of G_n .

Obviously we can choose $L_1 = K$. Now we would like to construct L_{n+1} out of L_n . Without loss of generality assume that $G_n - L_n$ does not contain any cycles. Otherwise take the graph obtained from L_n by adding cycles as long as there is a cycle in G_n whose edge set is contained in $G_n - L_n$.

Let L_{n+1}^0 be the subdivision of L_n contained in G_{n+1} , that is, L_{n+1}^0 is the graph obtained from L_n by subdividing every edge e of G'_n by adding $p_k(e)$ inner vertices. The definitions of G'_n and $p(e)$ are the same as in the construction of the graph G_{n+1} . Clearly, this is a connected subgraph of G_{n+1} spanning all of V_{n-1} in which all degrees are even.

So all we need to do is include the remaining vertices without making any degree odd or disconnecting the graph. We will now define a sequence L_{n+1}^i of subgraphs of G_{n+1} such that each L_{n+1}^i is connected and has only vertices with even degree.

Denote by \mathcal{H} the set of all copies of H which have been added when we constructed G_{n+1} out of G_n . For $H' \in \mathcal{H}$ let $S(H')$ be the corresponding copy of S . We call $H' \in \mathcal{H}$ *pending*, if $H' \cap L_{n+1}^i = \emptyset$. In this case we also call vertices of $S(H')$ *pending*.

Let F_{n+1} be the subdivision of $G_n - L_n$ in G_{n+1} . Note that F_{n+1} must be a forest because we assumed $G_n - L_n$ to be acyclic. Throughout the construction we will only make changes to L_{n+1}^i in F_{n+1} and in copies $H' \in \mathcal{H}$. In particular L_{n+1}^0 will be a subgraph of each L_{n+1}^i .

Call $H' \in \mathcal{H}$ *unsettled* if every vertex of $H' \cap L_{n+1}^i$ lies on a path in F_{n+1} connecting two pending vertices. In this case we also call vertices of $S(H')$

unsettled. Call a vertex *settled* if it is not unsettled. Notice that every pending vertex is unsettled while the converse need not necessarily be true.

Also notice that every vertex not contained in F_{n+1} must be settled. This immediately implies that for any unsettled H' and any two vertices in $S(H')$ there must be a path in F_{n+1} connecting the two.

Denote by F_{n+1}^i the convex hull of the set of unsettled vertices in F_{n+1} , that is, F_{n+1}^i is the subgraph of F_{n+1} induced by all unsettled vertices and all vertices which lie on a path between two unsettled vertices in F_{n+1} .

To construct L_{n+1}^{i+1} out of L_{n+1}^i consider a leaf v of F_{n+1}^i . Clearly, this is an unsettled vertex.

First we claim that $v \notin L_{n+1}^i$. Indeed, if v was contained in L_{n+1}^i then it would have to lie on a path between two pending vertices. Since any pending vertex is unsettled this would be a path in F_{n+1}^i , hence v would not be a leaf.

There is some $H' \in \mathcal{H}$ such that $v \in H'$. Choose any other vertex $u \in S(H')$ and let P be the unique u - v -path in F_{n+1} . Then the graph L_{n-1}^{i+1} is obtained from L_{n-1}^i by flipping all edges along P and adding a copy of the graph K_{uv} in H' .

Clearly this makes v a settled vertex. Since there is no change outside of $F_{n+1}^i \cup H'$ it follows that F_{n+1}^{i+1} has strictly less vertices than F_{n+1}^i . Furthermore it is easy to see that every vertex in L_{n+1}^{i+1} has even degree.

To show that L_{n+1}^{i+1} is connected we need the following definition. A *pivot vertex* is a vertex of F_{n+1} which is incident to an edge in L_{n+1}^i and an edge which is not contained in L_{n+1}^i . We now claim that every pivot vertex is connected to some vertex of G_n by a path in $L_{n+1}^i - F_{n+1}^i$.

This is true for L_{n+1}^0 because the only pivot vertices were already contained in G_n . Inductively assume that it was true for L_{n+1}^i . Since no changes are made to $L_{n+1}^i - F_{n+1}^i$ it is clearly still true for every pivot vertex of F_{n+1}^i after the modifications— $L_{n+1}^i - F_{n+1}^i \subseteq L_{n+1}^{i+1} - F_{n+1}^{i+1}$, so any path in the former graph is also a path in the latter graph.

The only possible pivot vertices in F_{n+1}^{i+1} which were not pivot vertices in F_{n+1}^i are u and v . By construction u is connected to v by a path outside of F_{n+1}^{i+1} . Now follow the u - v -path in F_{n+1} starting at v . If we reach some vertex of G_n before the first unsettled vertex we are done. Otherwise the first unsettled vertex must be a leaf in F_{n+1}^{i+1} . This implies that it is not contained in L_{n+1}^{i+1} , hence we must have passed at least one pivot vertex along the way. The path from v to the first pivot vertex v' that we passed is completely contained in $L_{n+1}^{i+1} - F_{n+1}^{i+1}$. It is an easy observation that all vertices on the v - v' -path are settled—an unsettled vertex on this path would imply an unsettled leaf of F_{n+1}^{i+1} on it which cannot be contained in L_{n+1}^{i+1} . Since v' is connected to some vertex of V_n by a path in $L_{n+1}^{i+1} - F_{n+1}^{i+1}$ this completes the proof of the claim.

To see that L_{n+1}^{i+1} is connected simply notice that every vertex of L_{n+1}^{i+1} is connected by a path in $L_{n+1}^{i+1} - F_{n+1}^{i+1}$ to either a vertex of G_n or a pivot vertex. The arguments are analogous to the arguments applied to the vertex v above.

We mentioned earlier that the vertex sets of the graphs F_{n+1}^i are strictly

decreasing in i . Since F_{n+1}^0 is finite this implies that we end up with an empty vertex set after finitely many steps. In particular, there is some i_0 such that $L_{n+1}^{i_0}$ has no unsettled and hence also no pending vertices. So every $H' \in \mathcal{H}$ contains some vertex of $L_{n+1}^{i_0}$.

Now let L_{n+1} be the graph obtained from $L_{n+1}^{i_0}$ by adding a copy of K in every $H' \in \mathcal{H}$ whose vertex set is not contained in $L_{n+1}^{i_0}$.

From the sequence L_n define a subgraph L of G by connecting every vertex in V_n to all of its neighbours in L_{n+1} . Clearly L contains an even number of edges in every finite cut of G . So we only need to show that L is end faithful in order to complete the proof of the proposition.

Choose two rays γ_1 and γ_2 in L belonging to the same end of G . Then γ_1 and γ_2 contain vertices v_1 and v_2 in the same component C of $G - V_n$. If we choose n large enough we may assume that those vertices both lie in V_{n+1} . By construction of L there is a path in $L[V_{n+2} \setminus V_n] = L_{n+2} - V_n$ connecting the two vertices. Repeating this argument yields infinitely many vertex disjoint paths in L connecting γ_1 to γ_2 .

Hence any two rays which are equivalent in G are also equivalent in L and thus L is end faithful. \square

The construction in the proof of Proposition 7.1 is rather specific to the example of Aharoni and Thomassen. However, the general approach may be useful in a more general setting:

- decompose the graph into finite parts,
- choose a “good” spanning subgraph in each of the parts, that is, one where the right vertices have odd degree,
- combine those subgraphs to obtain an end faithful spanning Eulerian subgraph,
- use Theorem 6.9 and Proposition 6.8.

In fact, it is easy to imagine a proof of Theorem 1.7 which follows the same lines. Lemma 5.8 delivers the decomposition. Once we have chosen the spanning trees in the finite parts we can define a spanning subgraph in each of the parts accordingly and combine those subgraphs to obtain an end faithful Eulerian spanning subgraph of G .

This suggests that it may be possible to use the same idea for a wider class of graphs satisfying the conditions of Georgakopoulos’ Conjecture 1.3. However, considering the amount of technical details needed in the proofs of Theorem 1.7 and Proposition 7.1 it is likely to be difficult.

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